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Money Creation in a Neoclassical Economy: Equilibrium Multiplicity and the Liquidity Trap*

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Abstract

We introduce banks that issue liquid deposits backed by illiquid bonds and capital into an otherwise standard cash-in-advance economy. Liquidity transformation can lead to multiple steady-state equilibria with different interest rates and real outcomes. Whenever multiple equilibria exist, one of them is a 'liquidity trap', in which nominal bond rates equal zero and banks are indifferent between holding bonds or reserves. Whether economic activity is higher in a liquidity trap or in a (coexisting) equilibrium with positive interest rates is ambiguous, but the liquidity trap equilibrium is more likely to go in hand with inefficient overinvestment. While liquidity transformation can lead to macroeconomic instability in the form of multiple equilibria, aggregate consumption is higher than in a cash-only economy, regardless of which equilibrium is selected.

Keywords: *Banks, Liquidity, Monetary Policy, Zero-Lower Bound.*

JEL codes: E4, E5.

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1 Introduction

In the years following the financial crisis of 2007/08, many advanced economies spent long periods in a liquidity trap—a situation in which returns on government bonds and other similar assets are at the zero lower bound and agents are indifferent between holding these assets or fiat money. This episode has inspired numerous papers aimed at better understanding the interaction between the financial system and the macroeconomy in general, and liquidity traps more specifically. The present paper is closely related, in particular, to the New Monetarist literature on the topic, which has significantly contributed to our understanding of liquidity traps (see, e.g. Williamson, 2012, 2016; Andolfatto and Williamson, 2015; Rocheteau et al., 2018; Altermatt, 2022). These papers highlight that interest rates on different assets have a liquidity premium component, with liquidity traps describing situations in which nominal rates on the most liquid assets are at the zero lower bound. A key insight from this literature is that liquidity traps can arise as a result of a 'shortage' of liquid assets, in which case the way out of the trap is to increase the supply of these assets, for instance, through open-market bond sales (Rocheteau et al., 2018).

In this paper, we place the insights on liquidity traps from the New Monetarist literature in a broader perspective and expand on them by studying liquidity transformation in a cash-in-advance (CIA) economy. In our model, households' consumption is subject to a liquidity constraint, and banks engage in liquidity transformation by issuing liquid deposits that are backed by illiquid assets. In this manner, assets that are themselves illiquid become 'indirectly' liquid, in the sense that banks can use these assets to back deposits. A liquidity trap then occurs when banks' demand for these indirectly-liquid assets pushes their return down to the zero lower bound.

Our main contributions in this paper are twofold. First, we develop a simple and clean setting based on a standard CIA model that allows to study the macroeconomic effects of money creation by banks and to analyse the root causes of liquidity traps. Key results from the New Monetarist literature on the emergence of liquidity traps arise naturally in our setup as well—for instance, we show that a liquidity trap is not generally the same as the Friedman rule. This demonstrates that these findings do not depend on the specific modelling structure in New Monetarist models. Second, we show how liquidity transformation can lead to multiple steady-state equilibria. These equilibria are characterised by different interest rates on indirectly-liquid assets, and whenever multiple equilibria exist, one of them constitutes a liquidity trap where the nominal rate on indirectly-liquid bonds is zero. Thus, we add to the existing monetary literature on liquidity traps by showing that a liquidity trap may not only arise due to a shortage of (directly or indirectly) liquid assets, but may also be the result of equilibrium selection.[1]

To derive these findings, we construct a neoclassical model in which firms use capital and labour to produce a good that can be used either for consumption or for capital accumulation. Following the CIA literature (Cooley and Hansen, 1989), households' consumption is subject to a liquidity constraint. The novelty of our framework lies in the fact that some households can use deposits instead of fiat money to pay for their consumption. Banks can create these liquid deposits by purchasing illiquid government bonds and capital, thereby engaging in liquidity transformation. While only deposits and fiat money are directly liquid – meaning that households can use them to purchase consumption goods – bonds and capital may still carry a liquidity premium due to their indirect liquidity value, which results from the fact that banks can finance these assets by issuing deposits.[2]

We first show that, relative to a cash-only economy à la Cooley and Hansen (1989), liquidity transformation increases aggregate consumption, investment and output via two channels. First, as is typical in economies with a liquidity-in-advance constraint, aggregate demand on the goods market depends on the average opportunity cost of carrying liquid funds, which declines when some households can pay with interest-bearing deposits. Second, when banks can finance some of their capital holdings with deposits, the liquidity premium on deposits will be partly passed through to capital, which increases investment and, as a consequence of the higher capital-labour ratio, real wages. The upshot is that liquidity transformation increases aggregate output and consumption relative to a cash-only economy, but it may also lead to inefficient overinvestment due to the liquidity premium on capital.

Three types of steady-state equilibria may exist when banks engage in liquidity transformation: a fundamental equilibrium, where liquidity is plentiful and thus capital is priced fundamentally; a zero-lower bound equilibrium, where liquidity is scarce and thus the bond interest rate is zero, and an interior equilibrium, where liquidity is scarce but not as scarce as in the zero-lower bound equilibrium case. We interpret the zero-lower bound equilibrium case as a liquidity trap. Here, it is

[1] Benhabib et al. (2001) show that active monetary policy adhering to the Taylor principle can also lead to multiple steady-state equilibria, with the liquidity trap equilibrium being one of them. In Benhabib et al. (2001), equilibrium multiplicity stems from self-fulfilling changes in inflation, which is different from the mechanism in our paper. Importantly, the liquidity trap equilibrium in our model is not associated with deflation.

[2] Aruoba et al. (2011) develop a New Monetarist model that shares some basic features with our model. Altermatt et al. (2024) introduce banks into the Aruoba et al. (2011) setup, and their model has many similarities with the one in the present paper. In addition to using a different micro-foundation for money, the two papers have different focuses: while Altermatt et al. (2024) study bank runs, the present paper focuses on steady-state equilibria instead.

important to highlight that in our model, a liquidity trap is not the same as the Friedman rule. Under the Friedman rule, the opportunity cost of carrying liquidity is zero. Meanwhile, a liquidity trap refers to a situation in which the interest rate differential between fiat money and indirectly-liquid bonds is zero. Rather than describing a situation where the opportunity cost of carrying liquidity is zero, a liquidity trap may thus represent a situation where liquidity is scarce overall and the liquidity premium on bonds equals the one on fiat money.

Next, we show that liquidity transformation can lead to multiple steady-state equilibria with different (bond) interest rates, i.e. that the three steady-state equilibria described above may co-exist for the same parameters. The underlying reason for this multiplicity is that households and firms interact both on the goods market and on the financial side of the economy (via banks). This dual interaction means that aggregate demand on the goods market and liquidity premia on indirectly-liquid assets influence each other: on the one hand, liquidity premia on bonds and capital affect the deposit rate, which determines households' opportunity costs of carrying liquid funds and thereby influences aggregate demand on the goods market; on the other hand, changes in aggregate demand impact investment and, thus, the supply of indirectly-liquid assets that banks can invest in to back deposit issuance, which, in turn, affects liquidity premia on these assets. This mechanism enables self-fulfilling prophecies in the real economy that are accompanied by changes in interest rates.

To gain some more intuition of why multiple equilibria are possible, consider first an equilibrium in which interest rates are high, i.e. liquidity premia on indirectly-liquid assets are low. In this equilibrium, firms expect strong demand in the goods market, which leads them to invest a lot despite the high interest rate. This large investment results in an ample supply of indirectly-liquid capital that banks can invest in to back deposits. Market clearing for indirectly-liquid assets then implies that liquidity premia on these assets are low (i.e. interest rates are high), which allows banks to pay high deposit rates. These high deposit rates mean that households' opportunity cost of carrying liquid funds is low, which leads to high demand in the goods market, thereby validating firms' initial expectation of strong demand. The mechanisms just described mean that self-fulfilling beliefs can also sustain a low-interest rate (liquidity trap) equilibrium. In this equilibrium, firms expect weak demand, and consequently, investment is low. The resulting scarcity of capital leads to high liquidity premia on indirectly-liquid assets and pushes their return to the zero-lower bound, such that banks are unable to pay interest on deposits. This means that households' opportunity cost of carrying liquid funds is high, which implies low demand in the goods market, thus validating firms' expectation of weak demand.[3]

The above description refers to a case where the liquidity trap equilibrium is characterised by lower investment, consumption and output than the equilibrium with a strictly positive interest rate. This need not be the case, however: if a liquidity trap coexists with an equilibrium in which the interest rate is strictly positive, it is in general ambiguous which equilibrium exhibits more economic activity. On the one hand, the opportunity cost of carrying liquid funds is lower in the equilibrium with a positive interest rate since banks pay strictly positive interest on deposits; taken by itself, this would imply higher economic activity in the high-interest rate equilibrium. On the other hand, the liquidity trap equilibrium features a higher liquidity premium on capital; keeping all else equal, this positively affects investment and real wages, but it also results in an inefficiently high capital-labour ratio. For all these reasons, there is no parameter-invariant welfare ranking of high- vs low-interest rate equilibria.

Finally, steady-state multiplicity does not generally imply that for a given initial capital stock, the economy can transition to different steady states. That is, even if there are multiple steady states, the dynamic equilibrium starting from any initial capital stock may still be unique. To address this, we study the transition dynamics of the model for parameters that induce multiple steady states. We find examples for both cases: sometimes, the economy can indeed transition to different steady states starting from some initial capital stock, while in other cases, the dynamic equilibrium is unique despite multiple steady states.

Related literature. Our paper is related to a broader macro-financial literature studying models in which assets other than fiat money carry a liquidity premium, either because these assets can be used directly to settle transactions (Lagos and Rocheteau, 2008; Andolfatto and Williamson, 2015; Rocheteau et al., 2018; Altermatt et al., 2023), can be sold against the ultimate means of payment swiftly and at low cost (Geromichalos and Herrenbrueck, 2022) or, as in our model, can be financed via the issuance of liquid bank deposits (Williamson, 2012, 2016; Altermatt, 2022; Keister and Sanches, 2023). In all these papers, aggregate demand in goods markets is positively related to the aggregate supply of assets that either directly or indirectly relax agents' liquidity constraints: the more abundant these assets, the lower the liquidity premia and, thus, the lower the opportunity cost of holding li-

[3]This is a slightly simplified description of the mechanism since demand on the goods market depends not only on the cost of carrying liquid funds but also on real wages, which in turn depend on firms' investment choices.

quid funds required to settle transactions. What distinguishes our paper is that the interaction between the supply of assets with liquidity value and aggregate demand goes in both directions, since aggregate demand influences firms' investment choices, which in turn affect the supply of indirectly-liquid capital. This mutual interaction between the aggregate supply of indirectly-liquid assets and aggregate demand is crucial for the multiplicity result we obtain.[4]

Furthermore, while monetary policy is not the main focus of this paper, it is instructive to relate our model to a recent theoretical literature on the transmission of monetary policy through banks. A number of recent papers have analysed how monetary policy affects the supply of deposits by monopolistic banks (Drechsler et al., 2017; Di Tella and Kurlat, 2021; Abadi et al., 2023; Wang, 2024). In these papers, an increase in the policy rate makes cash holdings less attractive relative to deposits, which raises banks' market power and leads to an increase in liquidity premia on deposits. Our model abstracts from market power, and it regards the interest rate on bonds with liquidity value for banks as the main policy rate; an increase in this rate allows banks to pay higher deposit rates, thus *decreasing* the liquidity premium on deposits. Another strand of the literature on the monetary transmission mechanism through banks focuses on banks' need to hold a liquidity buffer to self-insure against random deposit outflows (Drechsler et al., 2018; Bianchi and Bigio, 2022; D'Avernas and Vandeweyer, 2024). While banks do not face uncertainty regarding deposit withdrawals in our paper, these models are similar to ours in that policy determines the supply of assets with liquidity value for banks. An increase in the supply of liquid government debt reduces liquidity premia, which lowers banks' cost of holding liquidity buffers to back deposits and may thus increase banks' credit supply to the real economy. In these models, liquid government debt and loans to the real economy are to some extent complements for banks. This is different from our paper, where public and private debt are always substitutes for banks, and where the expansionary effect of a fall in liquidity premia stems from an increase in consumption demand by liquidity-constrained households.

Finally, our work fits into a broader literature on equilibrium multiplicity in macro models (see Toda and Walsh (2024) for a recent survey). Our findings relate in particular to a literature on multiple equilibria in Bewley-Huggett-Aiyagari

[4]Similar mechanisms are also present in Altermatt et al. (2024) but are not the subject of analysis in that paper. The only other paper featuring this two-way interaction between the supply of assets with liquidity value and aggregate demand that we are aware of is Geromichalos and Herrenbrueck (2022); they do not address the question of equilibrium multiplicity, and their assumption of a fixed labour supply suppresses some of the general equilibrium effects of liquidity transformation (for the latter, see the discussion in Subsection 5.2.2).

models: Toda (2017) documents that multiple stationary equilibria may exist in a Huggett (1993) model, and Açıkgöz (2018) shows the same for an Aiyagari (1994) economy. In these models, as in ours, the interest rate is the key equilibrium object, and multiplicity stems from the fact that the asset market may clear at different interest rates. Interestingly, although our framework is very different, we find conditions for multiplicity similar to those in an Aiyagari model, namely that multiple equilibria are more likely under a high rate of relative risk aversion and a low elasticity of substitution between capital and labour (Light, 2020).

Outline. The rest of this paper is structured as follows: Section 2 presents the environment; Section 3 discusses the equilibrium in an economy without banks; Section 4 introduces banks; Section 5 discusses the banking equilibrium, including cases where multiple steady-state equilibria coexist; finally, Section 6 concludes.

2 Environment

Time is discrete, indexed by $t = 0, 1, 2, ..$, and continues forever. To ease notation, we will mostly omit time subscripts, and we will use subscripts -1 and $+1$ to denote previous-period and next-period variables, respectively.

The economy is populated by a unit mass of infinitely-lived households. There is a single good in the economy, which can be consumed by households or converted into capital one for one. The good is produced according to

$$Y = F(K_{-1}, L), \tag{1}$$

where K_{-1} is capital brought into the current period, L is current aggregate labour supply, and F is a constant returns to scale (CRS) production function with the usual neoclassical properties. The aggregate resource constraint is

$$C + K = Y + (1 - \delta)K_{-1}, \tag{2}$$

where C is the aggregate consumption of households, and $\delta \in (0, 1]$ is the depreciation rate of capital. Both C and K are subject to nonnegativity constraints. Output is produced by a representative firm, which rents capital and labour from households at real prices ψ and w, respectively. We define the capital-labour ratio as $\kappa \equiv K_{-1}/L$, and $f(\kappa) \equiv F(\kappa, 1)$. Then, CRS of (1) implies

$$\psi = f'(\kappa), \tag{3}$$

$$w = f(\kappa) - \kappa f'(\kappa), \tag{4}$$

and zero profits for the firm.

In addition to capital K, two other storable objects exist in the economy: fiat money M and one-period nominal bonds B, both issued by the government. Households may store wealth in M, B, or K. Let ϕ denote the value of money (in terms of numeraire Y) and let i denote the market-clearing net nominal interest rate on bonds.[5] The gross inflation rate is denoted by $1 + \pi \equiv \phi_{-1}/\phi$.

Households' lifetime preferences are

$$U = \sum_{t=0}^{\infty} \beta^t \Big[u(c_t) - l_t \Big], \tag{5}$$

where $\beta \in (0, 1)$ denotes the households' fundamental discount factor, while c_t and l_t denote period-t household consumption and hours worked, respectively.[6] We assume $u'(c) > 0 > u''(c)$, $\lim\limits_{c \to 0} u'(c) = \infty$ and $\lim\limits_{c \to \infty} u'(c) = 0$. Households are subject to the budget constraint

$$c + k + \phi(m + b) \leq \big[\psi + (1 - \delta)\big] k_{-1} + wl + \phi\big(m_{-1} + (1 + i)b_{-1} + \tau\big),$$

where (m, b, k) denotes a household's portfolio of money, bonds, and capital, and τ is a nominal lump-sum transfer from the government (or tax if negative). Further, as is standard in the CIA literature, we assume that households are subject to the liquidity constraint

$$c \leq \phi m_{-1}, \tag{6}$$

i.e. consumption must be financed with money carried over from the last period.[7] For now, households can only pay with fiat money M. In Section 4, we will introduce banks and assume that some households can pay with bank deposits, which

[5] The value of money ϕ denotes how many units of numeraire Y one unit of money buys, meaning that $1/\phi$ is the price level.

[6] We use this quasilinear utility function because it allows to illustrate the key mechanisms of the model in the most transparent way. Assuming nonlinear disutility of labour poses no particular difficulties in our setup, but it will add some additional general equilibrium effects that are arguably not central to the main themes of our paper. We briefly discuss the case with nonlinear labour disutility in Appendix A.1.

[7] We adopt the timing convention that the goods market opens before the asset market. That is, after production has taken place at the start of a given period, households first use their liquid assets brought over from the previous period to purchase the produced consumption goods and then decide on the asset portfolio which they carry over to next period. An illiquid asset purchased at date t pays out in the asset market at date $t + 1$. Date-t wages and transfers are paid out in the asset market at date t.

need to be backed by bonds B and capital K.

There is a government issuing money M and nominal government bonds B. We denote by $\mathcal{M} \equiv \phi M$ and $\mathcal{B} \equiv \phi B$ the real supply of money and government bonds, respectively. The government lets the money supply grow at a constant rate γ,

$$M = (1+\gamma)M_{-1}, \tag{7}$$

where we assume $\gamma > \beta - 1$, which implies that steady-state inflation will be above the Friedman rule. Next, the government determines the real quantity of bonds as a function of output, the real money supply and the real interest rate,

$$\phi B = \mathcal{B}(\mathcal{M}, Y, i_{+1}, \pi_{+1}), \tag{8}$$

where $\mathcal{B}$ is some nonnegative, differentiable function. We leave the exact fiscal rule $\mathcal{B}(\cdot)$ open for the moment. The government budget constraint then writes

$$\phi(M+B) = \phi(M_{-1} + (1+i)B_{-1} + \tau), \tag{9}$$

where the transfer τ adjusts such that (9) holds given (7) and (8).

The first-best allocation. As a benchmark, consider the first-best allocation, which maximises households' lifetime utility (5) subject to the economy's resource constraint (1)-(2), $c_t = C_t$, $l_t = L_t$, and given some initial capital stock. The first-best allocation satisfies

$$u'(c_t) = \frac{1}{F_L'(K_{t-1}, L_t)} \quad \text{and} \quad u'(c_t) = \beta u'(c_{t+1})(F_K(K_{t+1}, L_t) + 1 - \delta).$$

Denoting κ^* and c^* as the first-best steady-state values of κ and c, respectively, we find that

$$f'(\kappa^*) + 1 - \delta = \frac{1}{\beta} \quad \text{and} \quad u'(c^*) = \frac{1}{w(\kappa^*)}, \tag{10}$$

where $w(\kappa^*) = f(\kappa^*) - f'(\kappa^*)\kappa^*$ is the real wage when $\kappa = \kappa^*$.

3 Equilibrium without Banks

In this section, we briefly discuss the equilibrium without banking, which serves as a useful reference point. All proofs are given in Appendix B.

In the unbanked economy, households choose their asset portfolio (m, b, k) each period. We can express the representative household's problem in recursive form as

$$V(m_{-1}, b_{-1}, k_{-1}) = \max_{c,l,m,b,k \geq 0} u(c) - l + \beta V(m, b, k)$$

subject to the constraints

$$c + k + \phi(m + b) \leq \big[\psi + (1 - \delta)\big]k_{-1} + wl + \phi\big(m_{-1} + (1 + i)b_{-1} + \tau\big), \qquad (11)$$

$$c \leq \phi m_{-1}, \qquad (12)$$

where (11) is the budget constraint and (12) is the CIA constraint.

Lemma 1. *In the equilibrium without banks, we have*

$$\frac{1+i}{1+\pi} = \psi + 1 - \delta = \frac{1}{\beta}\frac{w}{w_{-1}}, \qquad (13)$$

$$u'(c) = \frac{1+i}{w}, \qquad (14)$$

and the CIA constraint (12) *binds if* $i > 0$.

Condition (13) states that the real return on government bonds and capital must equal the rate of time preference times real wage growth. We call the corresponding nominal interest rate the 'Fisher rate' and denote it by

$$1 + \iota \equiv \frac{1+\pi}{\beta}\frac{w}{w_{-1}}. \qquad (15)$$

The Fisher rate is the nominal interest rate on a completely illiquid asset, i.e. an asset that can only be used to shift consumption across time but not to relax households' liquidity constraint. Quasilinear preferences imply that households (i) are willing to hold an arbitrary amount of assets paying the Fisher rate; (ii) are not willing to hold an asset paying less than the Fisher rate unless holding the asset relaxes the liquidity constraint; and (iii) would choose to hold an infinite amount of assets paying more than the Fisher rate.

Condition (14) pins down equilibrium consumption as a function of i and w; it shows that the nominal rate i (the opportunity cost of holding money) creates a wedge between the marginal utility of consumption and the opportunity cost of leisure.

For the remainder of this section, we impose steady state where all real variables are constant. Constant real money supply $\mathcal{M}$ implies $\gamma = \pi$, i.e. steady-state inflation equals the money growth rate. Since real wages are constant in the steady state, we obtain from (13) and (15) that

$$1 + i = 1 + \iota = \frac{1+\gamma}{\beta}. \qquad (16)$$

Given our assumption that $\gamma > \beta - 1$, we have from (16) that $i > 0$, which implies that the CIA constraint (12) binds. Next, from (3) and (13), we obtain that κ is pinned down by

$$\frac{1}{\beta} = f'(\kappa) + 1 - \delta. \qquad (17)$$

Finally, from (4) and (14), we have that steady-state consumption is determined by

$$u'(c) = \frac{1+i}{w} = \frac{1+i}{f(\kappa) - \kappa f'(\kappa)}. \tag{18}$$

Definition 1. *A steady state in the economy without banking is given by i, κ, c that solve equations* (16)-(18).

Proposition 1. *There exists a unique steady state in the economy without banking.*

We have from (10) and (17)-(18) that $\kappa = \kappa^*$ and $c < c^*$ in the steady-state equilibrium of the unbanked economy. Steady-state consumption is below the first-best level as a result of the CIA friction. Notice that, while κ is at its first-best level, the fact that aggregate consumption C (and thus aggregate output Y) is below the first best implies that the aggregate capital stock K is below the first best as well. Note also that the real bond supply $\mathcal{B}$ has no effect on the steady-state equilibrium in the unbanked economy. Policy only matters through the money growth rate, which determines the opportunity cost of holding money and, thus, affects equilibrium consumption, output and the aggregate capital stock.

The version of the model presented in this section represents the standard way of thinking about monetary policy in much of the monetary literature. In particular, there is no difference between the bond rate i and the Fisher rate ι. This, in turn, implies that a zero-lower bound equilibrium ($i = 0$) is equivalent to running the Friedman rule ($\iota = 0$) and delivers the first best. In the next section, we demonstrate why this way of thinking may be misleading when bonds (and possibly capital) carry a liquidity premium.

4 Banking Equilibrium

We now introduce a representative bank that issues nominal bank deposits D, which can be backed by government bonds B, capital K, and money M.[8] In any given period, some, but not all, households can pay for their consumption expenditures

[8]The role of banking introduced here, namely, to transform illiquid assets B and K into liquid deposits, is the same as in Altermatt (2022). While our results hinge on bonds and capital having liquidity value in the sense that they can be used (directly or indirectly) to pay for consumption, the precise way that these assets become liquid is not crucial. While we believe that liquidity transformation by banks is a natural way to model this, similar results would be obtained if capital and bonds were directly liquid instead, as, e.g. in Altermatt et al. (2023), or if they could be traded on secondary asset markets, as in Geromichalos and Herrenbrueck (2022).

with bank deposits. Specifically, with banking, households' liquidity constraint is given by

$$c \leq (1-\Theta)\phi m_{-1} + \Theta\phi d_{-1},$$

where d_{-1} are bank deposits carried over from the previous period, and $\Theta \in \{0,1\}$ is an i.i.d. idiosyncratic shock. In any given period, an individual household is in state $\Theta = 1$ with probability $\theta \in (0,1)$, in which case the household can pay for its consumption by transferring bank deposits.[9] With probability $1-\theta$, a household is in state $\Theta = 0$ and can only pay for consumption with fiat money. We introduce this uncertainty regarding individual liquidity needs because it allows the households' problem to be cast as an optimisation problem of a benevolent representative bank (or banking coalition), which simplifies the exposition.[10]

In each period, all households contribute an identical amount of funds to the bank, which the bank then invests in money, bonds and capital. At the start of the next period, when households' idiosyncratic states are realised, the bank provides a certain amount of money to households in state $\Theta = 0$ (think of this as households in state $\Theta = 0$ withdrawing money from the bank) while it provides a certain amount of bank deposits (backed by bonds, capital and possibly money) to households in state $\Theta = 1$. Since there is no aggregate uncertainty, by a law of large numbers, in any given period a fraction θ of the bank's depositors will be in state $\Theta = 1$ while a fraction $1-\theta$ will be in state $\Theta = 0$. To abstract from bank run equilibria, we assume that the bank observes the realisation of the states Θ.[11]

We denote c_Θ as consumption of households in state $\Theta \in \{0,1\}$, such that aggregate consumption equals

$$C = (1-\theta)c_0 + \theta c_1. \tag{19}$$

In each period, the bank chooses its asset portfolio (m, b, k) and the payouts given to depositors in state $\Theta \in \{0,1\}$ to maximise the expected utility of its depositors. The bank's problem can be expressed in recursive form as

$$V(m_{-1}, b_{-1}, k_{-1}) = \max_{c_0, c_1, l, m, b, k \geq 0} \theta u(c_1) + (1-\theta)u(c_0) - l + \beta V(m, b, k)$$

[9] The assumption that households in state $\Theta = 1$ only pay with deposits is without loss of generality since paying with deposits weakly dominates paying with fiat money.

[10] Alternatively, we could assume that some fixed set of households always pay with deposits and the remaining households always pay with fiat money. This would mean that hours worked are different for households that pay with deposits compared to those that pay with fiat money; among other things, hours worked would depend on the distribution of wealth between the two types of households.

[11] In Altermatt et al. (2024), we show how self-fulfilling panics can occur in a similar model.

subject to the constraints:

$$\begin{aligned}
&\theta c_1 + (1-\theta)c_0 + k + \phi(m+b) \\
&\qquad \le \big[\psi + (1-\delta)\big]k_{-1} + wl + \phi\big(m_{-1} + (1+i)b_{-1} + \tau\big), \qquad (20)\\
&(1-\theta)c_0 \le \phi m_{-1}, \qquad (21)\\
&\theta c_1 \le \phi d_{-1} = (1+i)\phi b_{-1} + \chi(\psi + 1 - \delta)k_{-1} + [\phi m_{-1} - (1-\theta)c_0]. \qquad (22)
\end{aligned}$$

Condition (20) is the budget constraint, which states that total consumption expenditure plus investment in money, bonds and capital cannot exceed the revenue from the previous asset portfolio plus households' income from wage payments and transfers.[12] Condition (21) is the liquidity constraint for households in state 0, which states that their consumption needs to be financed entirely with money. Finally, (22) is the liquidity constraint for households in state 1, which states that their consumption needs to be financed with deposits; the deposits, in turn, need to be backed by bonds, capital, and money not paid out to households in state 0 (think of the latter as excess reserves). The parameter $\chi \in [0,1]$ denotes the fraction of the bank's capital holdings that can be used to back deposits. Put differently, when deposits are backed by capital, the bank needs to hold $(1/\chi)$ units of capital per issued deposit, where we take the value of χ as exogenous. The remaining share $1-\chi$ of the capital can be regarded as being financed with illiquid bank equity held in equal proportion by all households. In what follows, we will sometimes say that capital is 'illiquid' when $\chi = 0$, i.e. when capital cannot be used to back deposits, and we will say that capital is 'liquid' whenever $\chi > 0$, keeping in mind that in our model, capital is only indirectly liquid via liquidity transformation.

Notice that the deposit return equals the return on the bank's asset portfolio used to back deposits. In this manner, the return from the bonds and (a fraction χ of) the capital brought into period t by the bank becomes (indirectly) available to finance period-t consumption of households that pay with deposits.

Lemma 2. *In the equilibrium with banks, consumption of households in state 0 and 1, respectively, satisfies*

$$u'(c_0) = \frac{1+\iota}{w} \quad \text{and} \quad u'(c_1) = \frac{1}{w}\frac{1+\iota}{1+i}, \qquad (23)$$

and the net return on capital satisfies

$$\psi + 1 - \delta = \frac{1}{\beta}\frac{w}{w_{-1}}\frac{1+i}{1+i+\chi(\iota - i)}. \qquad (24)$$

The liquidity constraint for households in state 0, (21)*, binds whenever $i > 0$, while the liquidity constraint for households in state 1,* (22)*, binds whenever $i < \iota$.*

[12] Since the bank maximises the expected utility of households, it is without loss of generality to assume that households contribute their entire income to the bank.

From (23), we have that equilibrium consumption of households in state 0 is determined by the real wage and the Fisher rate, while equilibrium consumption of households in state 1 is determined by the real wage and the ratio of the bond rate i to the Fisher rate ι. As in the economy without banks, the Fisher rate captures the opportunity cost of holding fiat money. The opportunity cost of holding indirectly-liquid bonds is determined by the ratio of the bond rate to the Fisher rate: the higher the bond rate i, the lower the opportunity cost of holding bonds and, hence, the lower the cost of providing liquidity to households in state 1. Whenever $i > 0$, we thus have $c_1 > c_0$. If $i = \iota$, then carrying bonds entails no opportunity cost, which means that providing liquidity to households in state 1 is costless and the associated liquidity constraint is slack. In contrast, when $i = 0$, the opportunity cost of carrying money and bonds is the same, such that the cost of providing liquidity to households in state 0 is the same as the cost of providing liquidity to households in state 1, which implies $c_0 = c_1$.

Next, equation (24) shows that if $i < \iota$ and $\chi > 0$, then capital carries a liquidity premium in the sense that the return on capital is lower than it would be if capital were not indirectly liquid (see (13)). The liquidity premium on capital is increasing in the share of capital that can be financed with deposits, χ, and decreasing in the bond rate i; the latter follows from the fact that banks need to be indifferent between holding bonds or capital.

We now impose steady state for the remainder of this section, which, as in Section 3, implies $\pi = \gamma$ and

$$1 + \iota = \frac{1+\gamma}{\beta}. \tag{25}$$

Combining (23)-(24) with the equilibrium factor prices $\psi = f'(\kappa)$ and $w(\kappa) = f(\kappa) - \kappa f'(\kappa)$, we obtain that in a steady state

$$u'(c_0) = \frac{1+\iota}{f(\kappa) - \kappa f'(\kappa)}, \tag{26}$$

$$u'(c_1) = \frac{1+\iota}{1+i}\frac{1}{f(\kappa) - \kappa f'(\kappa)}, \tag{27}$$

$$\frac{1}{\beta} = (f'(\kappa) + 1 - \delta)\left(1 + \chi\left(\frac{\iota - i}{1+i}\right)\right). \tag{28}$$

Consider first equation (28), which pins down κ. If $\chi = 0$ (i.e. capital cannot be used to back deposits), then there is no liquidity premium on capital, meaning that the return on capital is pinned down by fundamental parameters β and δ, and we have $\kappa = \kappa^*$. The same is true if $i = \iota$ (regardless of the value of χ), since the liquidity constraint for households in state 1 is slack in this case. Finally, if $\chi > 0$ and $i < \iota$, then indirectly-liquid assets are scarce, capital carries a liquidity premium, and we have $\kappa > \kappa^*$. In this case, κ is strictly decreasing in i, which results from a no-arbitrage condition: when i increases, the real return on bonds increases, which means that the real return on capital must increase as well.

Furthermore, from equations (26)-(27), we see that changes in i can indirectly affect both c_0 and c_1 by affecting κ and thus the real wage w. Specifically, as long as $\chi > 0$, an increase in i decreases κ, which then reduces w, thus negatively affecting both c_0 and c_1. An increase in i will therefore negatively affect c_0, while the effect on c_1 is ambiguous: on the one hand, the real wage falls, which as such has a negative effect on c_1, but on the other hand, the opportunity cost of providing liquidity to households in state 1 falls as well, which as such has a positive effect on c_1.

Different from the economy without banks, the equilibrium bond rate i is not fixed at the Fisher rate ι but will be determined by market clearing for indirectly-liquid assets. From the liquidity constraint of households in state 1, (22), we have that banks' aggregate demand for real assets $(\mathcal{M}, \mathcal{B}, K)$ in a steady state satisfies

$$\frac{1}{\beta(1+\iota)}\Big[\mathcal{M} + (1+i)\mathcal{B}\Big] + \chi\big(f'(\kappa) + 1 - \delta\big)K \geq \theta c_1 + (1-\theta)c_0 \quad \text{with equality if } i < \iota. \tag{29}$$

Furthermore, from the liquidity constraint of households in state 0, (21), we have that banks' demand for real money holdings in a steady state satisfies

$$\mathcal{M} \geq \beta(1+\iota)(1-\theta)c_0 \quad \text{with equality if } i > 0. \tag{30}$$

The supply of indirectly-liquid assets consists of the bond supply (8) plus the aggregate capital stock $K = \kappa L$. From the aggregate production function and resource constraint, (1)-(2), together with our expression for aggregate consumption (19), we obtain that aggregate hours worked in a steady state equal

$$L = \frac{(1-\theta)c_0 + \theta c_1}{f(\kappa) - \delta\kappa}, \tag{31}$$

such that the aggregate capital stock equals

$$K = \frac{\kappa}{f(\kappa) - \delta\kappa}\left[(1-\theta)c_0 + \theta c_1\right]. \tag{32}$$

Definition 2. *A steady-state equilibrium with banking is given by* c_1, c_0, κ, ι, $i \in [0, \iota]$, $\mathcal{M}$, $\mathcal{B}$, *and* K *that satisfy* (8), (25)-(30) *and* (32).

In the following, we will group steady-state equilibria into three cases depending on the equilibrium interest rate i:

Fundamental equilibrium (FE). An FE is a steady-state equilibrium where $i = \iota$, which implies that the liquidity constraint for households in state 0 binds while the one for households in state 1 is slack. The liquidity constraint for households in state 1 is slack since acquiring bonds to back deposits entails no opportunity cost for the bank. From (28), we have that in an FE, $1/\beta = f'(\kappa)+1-\delta$, which means that capital is fundamentally priced (i.e. it does not carry a liquidity premium), and $\kappa = \kappa^*$. Denoting c^{NB} as steady-state consumption in the unbanked economy, we have from (26)-(27) that consumption levels c_0 and c_1 in an FE satisfy $c^{NB} = c_0 < c_1 = c^*$.

Zero-lower bound equilibrium (ZE). A ZE is a steady-state equilibrium where $i = 0$. From (28), we have that in a ZE, $1/\beta = (f'(\kappa) + 1 - \delta)(1 + \chi\iota)$. The term $(1 + \chi\iota)$ reflects that, as long as $\chi > 0$, the bank overinvests in capital to provide liquidity to households, which implies that capital is priced above its fundamental value, and we have $\kappa > \kappa^*$. From (26)-(27), we have that $c_0 = c_1$ in a ZE, as the cost of providing liquidity is the same for all households. For a given κ, consumption levels in the ZE are the same as in the unbanked economy. However, as long as $\chi > 0$, κ is higher than in the unbanked economy, which implies that the real wage w and thus consumption levels c_0 and c_1 are higher as well. Note also that the ZE is the only equilibrium in which the bank may choose to hold excess reserves, i.e. the bank may hold more money than necessary to pay for consumption of households in state 0.

Interior equilibrium (IE). Finally, an IE is a steady-state equilibrium where $i \in (0, \iota)$. In an IE, providing liquidity to households in state 1 is costly for the bank since the real bond rate is below the discount rate; providing liquidity to households in state 0 (by holding non-interest bearing money) is even more costly. From (26)-(28), we have that $c_1 > c_0$ in an IE, which reflects that providing liquidity to households in state 0 is more costly than providing it to those in state 1. Furthermore, as long as $\chi > 0$, we have $\kappa > \kappa^*$ and $c_0 > c^{NB}$, meaning that consumption of all households is higher than in the unbanked economy.

In Section 5 below, we will discuss the conditions under which the different equilibrium cases (co-)exist. But before doing so, we want to take stock of some of the results derived in this section. The discussion above has shown that, compared to the allocation without banking, liquidity transformation affects the economy via two main channels. First, the transformation of illiquid bonds and capital into liquid deposits relaxes the liquidity constraint of households that pay with deposits, which has a positive effect on equilibrium consumption of these households. Second, when capital carries a liquidity premium, the capital-labour ratio κ increases, which in turn increases real wages and positively affects consumption of all households, including those that pay with money. For these two reasons, aggregate consumption C, aggregate output Y and the aggregate capital stock K are all higher in the steady-state equilibrium with banking compared to the unbanked economy. Note, however, that from (28), we have that $f'(\kappa) < \delta$ when χ is sufficiently high and i is sufficiently low, meaning that capital accumulation can be inefficiently high in the economy with banking.[13]

Furthermore, we have seen that with liquidity transformation, it is important to distinguish between the bond rate i and the Fisher rate ι, while these two rates are equivalent in the economy without banking (see the discussion at the end of Section 3). Recall that the Fisher rate equals the equilibrium interest rate on a completely illiquid asset, i.e. an asset that is neither directly liquid nor can it become indirectly liquid via liquidity transformation. Liquidity transformation by banks can lead to a liquidity premium on indirectly-liquid bonds and capital, which means that the bond rate may differ from the Fisher rate. This implies in particular that a zero-lower bound equilibrium ($i = 0$) is generally not equivalent to a Friedman rule equilibrium ($\iota = 0$). The Friedman rule implies that the economy must be at the zero-lower bound, but the converse is not true: a zero-lower bound equilibrium is any situation where $i = 0$, while ι may be strictly positive. While monetary policy is not the main focus of this paper, we will show in the following section that i can be varied by changing the real amount of bonds in circulation, $\mathcal{B}$. Since one can think of $\mathcal{B}$ as being affected by monetary policy interventions such as open market operations, we view i as the policy rate, while ι depends on the long-run inflation target γ.[14] For a more in-depth discussion of how to interpret i and ι in models similar to the one presented here and what this implies for various puzzles in the literature, see Herrenbrueck and Wang (2023).

[13] The result that there can be overinvestment when capital has liquidity value is well known in the New Monetarist literature (Lagos and Rocheteau, 2008). To the best of our knowledge, we are the first to show this result in a CIA model.

[14] Additional reasons to regard i and not ι as the rate set by monetary policy are that (i) i is observable in reality while ι typically is not, since almost all assets have some degree of liquidity; and (ii) while bond interest rates are not exactly identical to policy rates in reality, actual policy rates such as the Fed funds rate behave similarly to bond rates.

5 (Co-)Existence of Equilibrium Cases

We now discuss the conditions under which the different equilibrium cases exist and whether there is a unique equilibrium. Uniqueness here means that given the economy's fundamental parameters as well as fiscal and monetary policies, we can determine which of the three equilibrium cases occurs. If there is coexistence of equilibrium cases, then there is no clear mapping from a given set of policies and parameters to equilibrium cases, and several real outcomes are possible for the same underlying economic conditions.

The equilibrium interest rate i will be such that the market for indirectly-liquid assets clears. To derive existence conditions for the different equilibrium cases, it will be useful to slightly reformulate the aggregate supply and demand for indirectly-liquid assets from Section 4. First, we denote

$$\mathcal{A}^s(i) \equiv \mathcal{B}(i) + \frac{\chi(1+\iota)}{1+i+\chi(\iota-i)} K(i) \tag{33}$$

as the 'bond-equivalent' supply of indirectly-liquid assets. That is, a given real bond supply $\mathcal{B}$ plus a capital stock K provide the same liquidity value to banks as a real bond supply of $\mathcal{A}^s$. For $\chi = 0$ (capital has no liquidity value), we have $\mathcal{A}^s = \mathcal{B}$, and for $\chi = 1$ (capital has the same liquidity value as bonds), we have $\mathcal{A}^s = \mathcal{B} + K$. For $\chi \in (0,1)$, we have $\mathcal{A}^s \in (\mathcal{B}, \mathcal{B}+K)$ with $\mathcal{A}^s$ increasing in χ. Note that the dependence of all variables on i is explicit in expression (33).[15]

According to (32), the aggregate capital stock K can be expressed as an increasing function of aggregate consumption C and the capital-labour ratio κ:

$$K(i) = q(\kappa(i))\, C(i), \quad \text{with} \quad q(\kappa) \equiv \frac{\kappa}{f(\kappa) - \delta\kappa}. \tag{34}$$

The expression $q(\kappa)$ denotes the ratio of aggregate capital K to aggregate consumption C, with $q'(\kappa) > 0$.[16] Notice that changes in the interest rate i affect K both via their effect on κ and via their effect on C. As long as $\chi > 0$, an increase in i reduces κ (see equation (28)), which taken by itself has a negative effect on K. As we have seen in Section 4, whether C increases or decreases in i is ambiguous: on the one hand, an increase in i reduces the real wage w (via the effect of i on κ), which negatively affects both c_0 and c_1; on the other hand, an increase in i reduces the cost of providing liquidity to households paying with deposits, which has a positive

[15]With regard to the real bond supply, the only thing that will matter for our purposes is how it changes with i; we thus express the real bond supply as a function of i only, capturing both direct and indirect (e.g. via $\mathcal{M}$ or Y) effects of i on $\mathcal{B}$.

[16]Denoting $\hat{\alpha}(\kappa) \equiv [\kappa f'(\kappa)]/f(\kappa)$ as the capital share (the fraction of output going to capital owners), we have $q(\kappa) = [\hat{\alpha}(\kappa)]/[f'(\kappa) - \delta\hat{\alpha}(\kappa)]$, i.e. q is increasing in the capital share.

effect on c_1. If C increases in i, then it is possible that K increases in i—as we will see, this property of our model facilitates the emergence of multiple equilibria.

Next, we denote

$$\mathcal{A}^d(i) \equiv \frac{\beta(1+\iota)}{1+i}\theta c_1(i) \tag{35}$$

as the 'bond-equivalent' demand for indirectly-liquid assets by banks. That is, if deposits were backed solely by bonds and no other assets, then banks would need to hold a real bond quantity $\mathcal{A}^d$ to back the deposits required to grant households in state 1 a given consumption level c_1.

Expression (35) highlights that, as usual, changes in i affect the demand for indirectly-liquid assets $\mathcal{A}^d$ both via a substitution effect and an income effect. On the one hand, an increase in i reduces the cost of providing liquidity to households in state 1, which reduces the effective cost of consumption for these households and thus increases the optimal consumption level c_1; taken by itself, this has a positive effect on $\mathcal{A}^d$. On the other hand, an increase in i means that the bank must purchase a smaller quantity of indirectly-liquid assets to provide households in state 1 with a given consumption level c_1, which taken by itself has a negative effect on $\mathcal{A}^d$. Which effect dominates depends on the curvature of the utility function $u(c)$. Note that, in addition to these standard substitution- and income effects, changes in i affect $\mathcal{A}^d$ also via their effect on the real wage, which in turn affects the desired consumption level c_1.

Using our definitions of $\mathcal{A}^s$ and $\mathcal{A}^d$ together with conditions (30) and (29), we obtain the following condition for market clearing for indirectly-liquid assets:

$$\mathcal{A}^s(i) \begin{cases} \geq \mathcal{A}^d(i) & \text{if } i = \iota \quad \text{(FE)} \\ = \mathcal{A}^d(i) & \text{if } i \in (0,\iota) \quad \text{(IE)} \\ \leq \mathcal{A}^d(i) & \text{if } i = 0 \quad \text{(ZE)} \end{cases} . \tag{36}$$

Condition (36) shows that the economy will be in an FE if the supply of indirectly-liquid assets is plentiful relative to the demand, it will be in a ZE if the supply of indirectly-liquid assets is scarce relative to the demand, and it will be in an IE in an intermediate case. Note that in a ZE, banks are indifferent between holding money, bonds or capital, and the scarcity of indirectly-liquid assets leads banks to hold excess reserves, i.e. they back some of their deposits with money. Conversely, in an FE, holding bonds and capital entails no opportunity cost for banks, such that banks are willing to hold more assets than required to back deposits.

We can see immediately from expression (36) that a necessary condition for multiple steady-state equilibria is that the difference $\mathcal{A}^s(i)-\mathcal{A}^d(i)$ (weakly) increases in i over at least part of the interval $i \in [0,\iota]$. Intuitively, if an increase in i (i.e.

a decrease in the price of indirectly-liquid assets) leads to an increase in the asset supply $\mathcal{A}^s$ relative to the asset demand $\mathcal{A}^d$, then changes in asset prices can be self-fulfilling and multiple equilibria are possible.

Note that the economy can always be put in an FE by saturating it with bonds. Furthermore, multiple equilibria become more likely when the real bond supply $\mathcal{B}(i)$ increases in i, which makes it more likely that $\mathcal{A}^s(i) - \mathcal{A}^d(i)$ increases in i. Our main interest is to study how the liquidity premium on capital can generate multiple equilibria. To focus on this, we will assume for most of the following analysis that the government keeps the real amount of bonds in circulation at some constant level. Additionally, we will also consider the case where the government keeps the ratio of money to total government debt constant, which, as we will show, implies that $\mathcal{B}(i)$ decreases in i. We will now briefly discuss these two policies.

Fixed real bond supply. Suppose the government keeps the real quantity of bonds at some (exogenous) constant level $\overline{\mathcal{B}} \geq 0$. With a constant real bond supply, changes in the aggregate supply of indirectly-liquid assets $\mathcal{A}^s$ are fully driven by changes in the aggregate capital stock K. It will be helpful to denote

$$Q(i) \equiv \mathcal{A}^d(i) - [\mathcal{A}^s(i) - \mathcal{B}(i)] = \mathcal{A}^d(i) - \frac{\chi(1+\iota)}{1+i+\chi(\iota - i)} K(i) \tag{37}$$

as the difference between banks' demand for indirectly-liquid assets to back deposits and the total capital stock that can be used to back deposits. With a fixed real bond supply $\overline{\mathcal{B}}$, we can then rewrite our asset market clearing condition from (36) as:

$$\overline{\mathcal{B}} \begin{cases} \geq Q(i) & \text{if } i = \iota \quad \text{(FE)} \\ = Q(i) & \text{if } i \in (0, \iota) \quad \text{(IE)} \\ \leq Q(i) & \text{if } i = 0 \quad \text{(ZE)} \end{cases} . \tag{38}$$

Lemma 3. *Suppose $\mathcal{B}(\cdot) = \overline{\mathcal{B}}$. A sufficient condition for an FE, an IE and a ZE to coexist for some $\overline{\mathcal{B}} > 0$ is that $Q(0) > 0$ and $Q'(i) < 0$ for $i \in [0, \iota]$.*

Intuitively, if $Q'(i) < 0$, then the amount of capital available to back deposits increases by more than banks' demand for indirectly-liquid assets when i increases. Put differently, a fall in asset prices (an increase in i) goes in hand with an increase in the asset supply relative to asset demand, such that the fall in asset prices can be self-fulfilling. Note that if $Q(0) < 0$, then capital is so plentiful that the economy cannot be in a ZE for any nonnegative bond supply.

Constant money-to-debt ratio. Suppose now that, instead of fixing the real bond supply, the government fixes the ratio of money to total nominal government debt.

Denoting $\eta \in (0,1]$ as the constant money-to-debt ratio, we then have $M = \eta(M+B)$ and $\mathcal{B} = [(1-\eta)/\eta]\mathcal{M}$. The key difference from before is that the real bond supply now changes with i since equilibrium real money balances depend on i. To see this, note that whenever $i > 0$, the CIA constraint for households in state 0 binds, and steady-state real money balances $\mathcal{M}$ are strictly increasing in c_0 (see (30)). From (26) we have that c_0, in turn, is strictly increasing in the real wage w, which itself is decreasing in i (via the effect of i on κ). Therefore, a higher steady-state interest rate i is associated with lower $\mathcal{M}$. Given that $\mathcal{B}$ is a fixed multiple of $\mathcal{M}$, a higher steady-state interest rate is thus associated with a lower $\mathcal{B}$. Via this channel, an increase in i exerts a negative effect on the aggregate asset supply $\mathcal{A}^s(i)$ under a fixed money-to-debt ratio, which makes it less likely that $\mathcal{A}^s(i) - \mathcal{A}^d(i)$ increases in i. For this reason, multiple equilibria are generally more difficult to obtain with a fixed money-to-debt ratio than with a fixed real bond supply:

Lemma 4. *Suppose an FE and a ZE coexist for some fixed money-to-debt ratio $\eta \in (0,1)$. Then, there exists some $\overline{\mathcal{B}} > 0$ such that an FE and a ZE coexist in the same economy when the real bond supply is held constant at $\overline{\mathcal{B}}$.*

5.1 Illiquid Capital

In this subsection, we briefly discuss the case where capital cannot be used to back deposits ($\chi = 0$). The net return to capital is then pinned down by β (see condition (28)), and we have $\kappa = \kappa^*$.

Proposition 2. *Suppose $\chi = 0$, $\dfrac{\partial \mathcal{B}(i)}{\partial i} \leq 0$ and $u(c) = \dfrac{c^{1-\sigma}}{1-\sigma}$. Then, a sufficient condition for uniqueness of the steady-state equilibrium is $\sigma < 1$.*

The intuition for the result in Proposition 2 is as follows. As described above, multiple equilibria are only possible if the asset supply $\mathcal{A}^s(i)$ increases relative to the asset demand $\mathcal{A}^d(i)$ when i increases. With $\chi = 0$, we have $\mathcal{A}^s(i) = \mathcal{B}(i)$. Therefore, given $\mathcal{B}'(i) \leq 0$, a necessary condition for multiplicity is that the demand for indirectly-liquid assets $\mathcal{A}^d(i)$ decreases in i, i.e. the quantity of assets demanded falls when asset prices fall. As described further above, whether $\mathcal{A}^d(i)$ increases or decreases in i depends on whether the substitution or the income effect dominates when i changes.[17] If σ is low, the substitution effect dominates and $\mathcal{A}^d(i)$ increases in i, which precludes multiple equilibria.

The result of Proposition 2 evidently applies to the case where the real bond supply $\mathcal{B}$ is held constant. Furthermore, it is not difficult to see that the result

[17] We show in the proof of Proposition 2 that a sufficient condition for uniqueness with a general utility function is $c_1'(i)/c_1(i) > 1/(1+i)$ over $i \in [0, \iota]$.

also applies to the case where the money-to-debt ratio is kept constant. The reason is that, as described above, a constant money-to-debt ratio implies that a higher steady-state interest rate i is associated with a lower real bond supply $\mathcal{B}$.

The following result states that with illiquid capital, multiple equilibria can occur when σ is high:

Proposition 3. *Suppose $\chi = 0$, $\mathcal{B}(\cdot) = \overline{\mathcal{B}}$ and $u(c) = \dfrac{c^{1-\sigma}}{1-\sigma}$ with $\sigma \geq 1$. Then, there exist debt levels $\overline{\mathcal{B}} > 0$ such that an FE, an IE and a ZE coexist.*

If σ is high, then households' optimal consumption level c_1 changes only slightly when i increases, implying that the income effect dominates and the demand for indirectly-liquid assets $\mathcal{A}^d(i)$ falls when i increases. In this case, a decrease in bond prices (i.e. an increase in i) goes in hand with a decrease in the real amount of bonds demanded, which can in turn justify the lower bond prices and opens up the possibility of multiple equilibria. Notice that, in a similar vein, an increase in inflation π can lead to an increase in the amount of real money $\mathcal{M}$ demanded when σ is high.

While we are not aware of a reference that makes this point explicitly, we believe the result that multiple steady-state equilibria may exist when $\sigma \geq 1$ is well understood among economists working on banking and macro-finance, since several papers from this literature assume $\sigma < 1$ (examples include Haslag and Martin (2007), Williamson (2012), and Altermatt (2022)).[18] We also focus on $\sigma < 1$ for the remainder of the paper because (i) we want to highlight how a liquidity premium on capital can be a novel source of equilibrium multiplicity, over and above the multiplicity that may result from a strong income effect due to high σ; and (ii) as mentioned above, assuming $\sigma > 1$ implies that money demand increases with inflation, which is contrary to the empirical evidence.

With illiquid capital and $\sigma < 1$, our economy behaves very similarly to the one in Williamson (2012).[19] Marginal exogenous changes in the real bond supply $\mathcal{B}$ have no effect on the steady-state equilibrium if bonds are very scarce (in which case the economy will be in a ZE) or plentiful (in which case the economy will be in an FE). If the economy is in an IE, a marginal exogenous increase in $\mathcal{B}$ will lead to an increase in i, thereby relaxing the liquidity constraint of households in state 1, which will

[18] Similarly, Toda and Walsh (2024) point out that $\sigma \leq 1$, with the associated dampening of income effects, typically guarantees equilibrium uniqueness in heterogeneous agent macro models, including OLG models and Bewley (1986) type models.

[19] The main difference is that in Williamson (2012), capital is not used to produce the consumption good sold against money and deposits. However, with illiquid capital, this difference does not materially affect the behaviour of the economy.

increase aggregate consumption, output and the capital stock.

5.2 Liquid Capital

We now consider the case where $\chi > 0$, meaning that capital can be used to back deposits. As shown above, if capital is indirectly liquid, then aggregate consumption and the supply of indirectly-liquid assets both influence each other: capital investment depends on aggregate consumption (see (34)), while aggregate consumption depends itself on the supply of indirectly-liquid assets, since the asset supply affects the market-clearing interest rate i (see (36)), which in turn determines deposit rates and, thus, households' opportunity cost of carrying liquid funds. This feedback loop between aggregate consumption and the supply of indirectly-liquid assets expands the set of parameters for which multiple equilibria are possible—in particular, multiple equilibria can occur for $\sigma < 1$:

Proposition 4. *Consider an economy with a utility function*

$$u(c) = D\frac{c^{1-\sigma}}{1-\sigma} \text{ with } \sigma < 1 \tag{39}$$

and a CES production function

$$Y = A\left(\alpha K_{-1}^{\frac{\rho-1}{\rho}} + (1-\alpha)L^{\frac{\rho-1}{\rho}}\right)^{\frac{\rho}{\rho-1}} \tag{40}$$

with $A, D > 0$, $\alpha \in (0,1)$ *and* $\rho \geq 0$. *Suppose* $\mathcal{B}(\cdot) = \overline{\mathcal{B}}$. *There exist parameters and debt levels* $\overline{\mathcal{B}} > 0$ *for which an FE, an IE, and a ZE coexist. Denoting* Y^j *as the aggregate output in equilibrium case* $j \in \{FE, IE, ZE\}$, *there exist cases with multiple equilibria where* $Y^{FE} > Y^{IE} > Y^{ZE}$, *and there exist cases with multiple equilibria where* $Y^{FE} < Y^{IE} < Y^{ZE}$.

We prove Proposition 4 with examples. Consider first the example shown in Figure 1. The panel at the top left shows demand and supply for indirectly-liquid assets, $\mathcal{A}^d(i)$ and $\mathcal{A}^s(i)$, for this economy. Comparing the graph with the asset market clearing condition (36) reveals that the conditions for all three types of equilibrium are satisfied simultaneously: we have $\mathcal{A}^d(0) > \mathcal{A}^s(0)$, which constitutes a ZE, since banks can make up the shortfall in asset supply by holding excess reserves at the zero-lower bound; we also have $\mathcal{A}^d(\iota) < \mathcal{A}^s(\iota)$, which constitutes an FE, since banks are willing to hold the excess supply of assets as they earn the Fisher rate and holding them is thus costless; finally, we have $\mathcal{A}^d(0.061) = \mathcal{A}^s(0.061)$, which implies that there is an IE at a bond rate of 6.1%. In this example, we have $Y^{FE} > Y^{IE} > Y^{ZE}$, so the FE is the *high-activity equilibrium* while the ZE is the *low-activity equilibrium.*

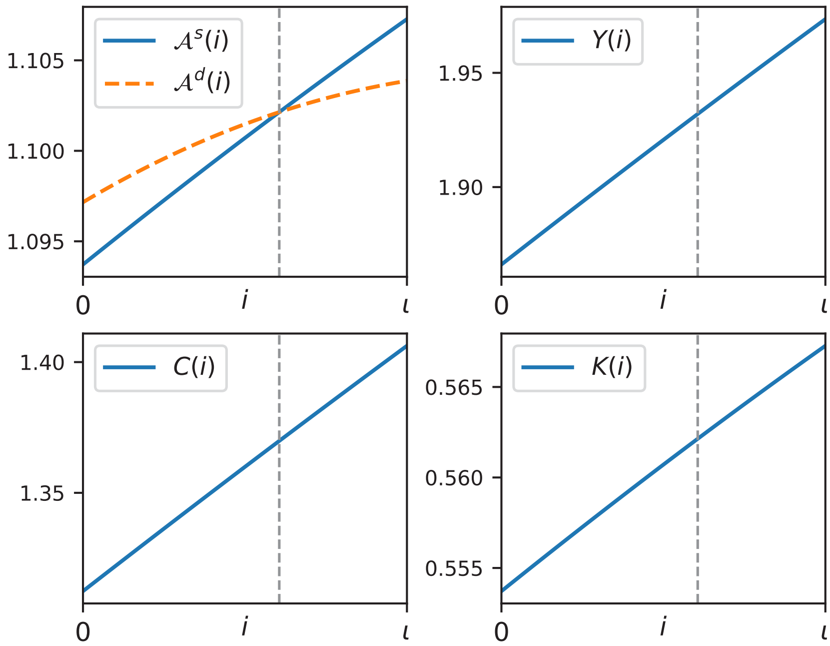

Figure 1: Example of multiple equilibria where the FE is the high-activity equilibrium. The parameters for this example are: $\beta = 0.95, \gamma = 0.045, \theta = 0.8, \sigma = 0.55, D = 1, A = 2, \alpha = 0.1, \rho = 1/3, \chi = 1, \delta = 1, \mathcal{B} = 0.54$. The Fisher rate equals $\iota = 0.1$.

Figure 1 highlights that the steady-state multiplicity in our model stems from multiple equilibria on the asset market. Since the market-clearing interest rate i affects the real side of the economy through firms' investment choices as well as households' choices of real balances (which in turn affect consumption), each asset-market equilibrium corresponds to a different real equilibrium. To understand the intuition for the multiplicity in more detail and why the FE is the *high-activity equilibrium* here, recall first that an increase in i has counteracting effects on aggregate consumption C: on the one hand, an increase in i implies lower opportunity costs of carrying liquidity for households who can pay with deposits, but on the other hand, an increase in i leads to a decrease in the capital-labour ratio κ and, thus, to a decrease in the real wage w. As the bottom left panel in Figure 1 shows, the first effect dominates in this example, such that we have $C'(i) > 0$. Next, remember that since the aggregate capital stock K depends both on C and on κ (see (34)), and since κ falls in i, K may increase or decrease in i when $C'(i) > 0$. As the lower right panel in Figure 1 shows, in this economy, K is increasing in i, which means that at higher i, firms are willing to invest more due to the higher demand for consumption. Then, $Y^{FE} > Y^{IE} > Y^{ZE}$ follows since with $\delta = 1$, we have $Y = C + K$. In this example, the willingness of firms to invest more despite higher interest rates is key for the multiplicity—it implies that both aggregate demand and aggregate supply in the goods market increase when i increases, which means that multiple equilibria with different output levels are possible.

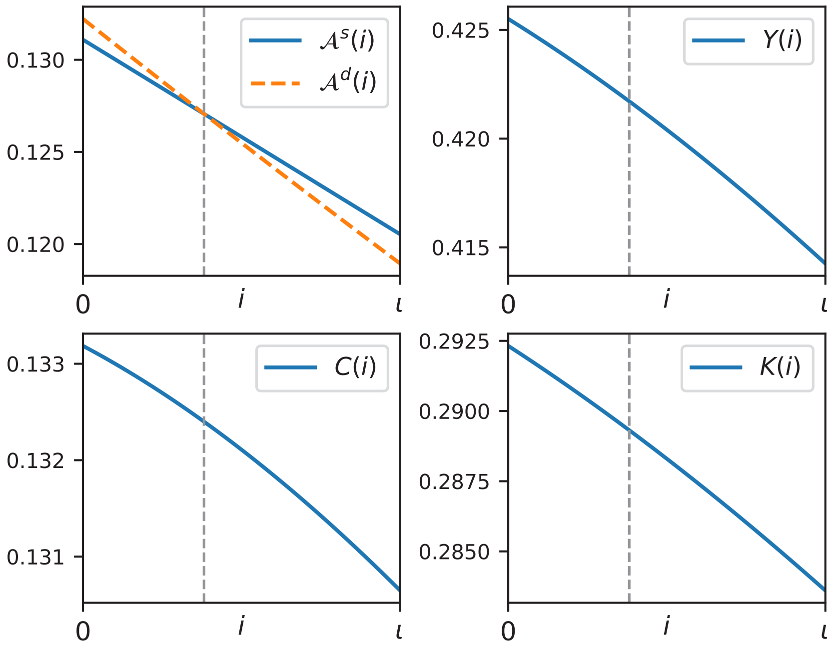

Figure 2: Example of multiple equilibria where the ZE is the high-activity equilibrium. The parameters for this example are: $\beta = 0.95, \gamma = 0.045, \theta = 0.95, \sigma = 0.5, D = 1.1, A = 1.3, \alpha = 0.2, \rho = 1/12, \chi = 0.425, \delta = 1, \mathcal{B} = 0$. The Fisher rate equals $\iota = 0.1$.

Consider next the example shown in Figure 2. The top left panel in Figure 2 shows $\mathcal{A}^d(i)$ and $\mathcal{A}^s(i)$ for this economy, and again the conditions for all types of equilibrium from (36) are satisfied simultaneously, with the IE existing for an interest rate of 3.8%. The bottom panels in Figure 2 depict $C(i)$ and $K(i)$ for this example. As we can see, we have $C'(i) < 0$, i.e. at higher i, the negative effect on C from lower real wages w dominates. Consequently, we also have $K'(i) < 0$, as $C'(i) > 0$ is a necessary condition for K to be increasing in i. Since both C and K decrease in i, it follows that $Y^{FE} < Y^{IE} < Y^{ZE}$. In this example, both aggregate demand and aggregate supply in the goods market decrease when i increases, which again enables the existence of multiple equilibria with different output levels, with the ZE being the *high-activity equilibrium*.

5.2.1 An Explicit Condition for Multiplicity

Deriving an explicit parameter condition that guarantees the existence of multiple equilibria is easier for a fixed money-to-debt ratio than for a fixed real bond supply:

Proposition 5. *Consider an economy with a CRRA utility function* (39) *with* $D = 1$*, a CES production function* (40) *with* $A = 1$*, and* $\chi > 0$*. Suppose* $\mathcal{B} = \frac{1-\eta}{\eta}\mathcal{M}$*,* *and define* $q^* \equiv q(\kappa^*)$*. If*

$$q^* < (\beta\theta)/\chi, \tag{41}$$

and

$$\rho < \frac{\beta f'(\kappa^*)}{q^*(1+\delta q^*)}\left[\frac{\theta}{\chi^2}\left(\beta - \frac{1}{\sigma}\Big(\beta - \chi q^*\Big)\right) - \left(\frac{1}{\chi} - 1\right) q^*\right], \tag{42}$$

and ι *is sufficiently close to 0, then there exist money-to-debt ratios* $\eta \in (0,1)$ *such that an FE, an IE and a ZE coexist.*

Condition (41) ensures that the first-best capital stock (more precisely, the share of the capital stock that can be used to back deposits) is not too high, since otherwise the economy will be in an FE for any nonnegative bond supply. Condition (42) is equivalent to the condition that $\mathcal{A}^s(i) - \mathcal{A}^d(i)$ strictly increases in i in an ϵ-neighbourhood of the Friedman rule ($\iota = 0$). While assuming the Fisher rate ι to be close to zero considerably simplifies the derivation of a parameter condition guaranteeing the existence of multiple equilibria, this is not a necessary condition for equilibrium multiplicity: in Appendix A.2, we provide an example where multiple equilibria exist for a constant money-to-debt ratio with $\iota = 0.1$.

Condition (42) sheds light on the parameter conditions that facilitate the emergence of multiple equilibria. First, we see that (42) can only be fulfilled if the elasticity of substitution between labour and capital, ρ, is not too high. A relatively low ρ ensures that the capital-labour ratio κ does not fall too much when i increases, which means that the aggregate capital stock K will not be affected too negatively by an increase in i. This makes it more likely that the supply of indirectly-liquid assets $\mathcal{A}^s$ increases relative to the demand $\mathcal{A}^d$ when i increases, which, as discussed further above, makes the existence of multiple equilibria more likely. Next, note that for any q^* satisfying (41), we have $\beta > \chi q^*$, which means that (42) is more likely to be fulfilled when σ is not too low. Indeed, a necessary condition for (42) to hold is that $\sigma > 1 - (\chi q^*)/\beta$. If σ is very low, households will strongly increase consumption in state 1 when i increases. As a result, an increase in i will go in hand with a large increase in $\mathcal{A}^d$ (see (35)), which makes it less likely that $\mathcal{A}^s$ will increase relative to $\mathcal{A}^d$ when i increases.[20] Furthermore, providing that σ is in a region conducive to multiplicity (i.e. not too low), condition (42) is more likely to be fulfilled when the share of households paying with deposits, θ, is higher. This is intuitive since the lower θ is, the closer we are to a cash-only economy, which has a unique equilibrium (see Section 3).

[20] Light (2020) find similar results for the Aiyagari (1994) model: equilibrium multiplicity is ruled out if $\sigma < 1$ and $\rho > 1$.

The result in Proposition 5 also allows to show formally that multiple equilibria can occur for $\sigma < 1$ under a constant money-to-debt ratio. To see this, note that for $\chi = 1$, condition (42) becomes

$$\rho < \frac{\beta\theta f'(\kappa^*)}{q^*(1+\delta q^*)}\left(\beta - \frac{1}{\sigma}(\beta - q^*)\right). \tag{43}$$

The right-hand side of condition (43) is strictly positive whenever $\sigma > 1-(q^*/\beta)$. It follows that for any σ satisfying $1-(q^*/\beta) < \sigma < 1$, condition (43) will be fulfilled as long as ρ is sufficiently low.

Finally, it follows immediately from Lemma 4 that the parameter conditions (41)-(42) are also sufficient to obtain multiplicity (in the vicinity of the Friedman rule) for some constant real bond supply.

5.2.2 Comparing Equilibria

One implication of our model is that, due to the possibility of multiple equilibria, controlling the amount of government bonds in circulation, e.g. through open-market operations, may not implement a unique interest rate i. Instead, the interest rate may need to be fixed more directly, for instance, by standing ready to buy/sell bonds at a given price. The natural next question to ask is which equilibrium a policymaker should aim to implement in case there are multiple equilibria. As we shall see, there is no general answer to which equilibrium is the most desirable one.

We will focus our discussion on the comparison between the FE and the ZE, and specifically on how these equilibria relate to the first-best steady-state allocation derived in Section 2.[21] Note that when comparing equilibrium allocations to the first best, there are three margins to consider: the quantity of output produced, the combination of capital and labour with which a given output is produced, and how a given output is distributed among households. Throughout this subsection, we will assume $\chi > 0$ and $u(c) = c^{1-\sigma}/(1-\sigma)$, and we will denote steady-state values in the FE (ZE) by an FE- (ZE-) superscript.

First, as shown in Section 4, we have $\kappa^{FE} = \kappa^*$ and $\kappa^{ZE} > \kappa^*$, i.e. the capital-labour ratio equals its first-best value in the FE, while it is higher than the first-best value in the ZE. Next, from (1), (2) and (10), we obtain that hours worked in the first best satisfy

$$l^*(\kappa) = \frac{c}{f(\kappa)-\delta\kappa} = \frac{[w(k)]^\sigma}{f(\kappa)-\delta\kappa}. \tag{44}$$

[21] Whenever both an FE and a ZE exist, then an IE (with κ in between the FE and the ZE) exists as well.

According to (26)-(27) and (31), hours worked in the ZE and the FE, respectively, satisfy

$$l^{ZE}(\kappa) = \left(\frac{1}{1+\iota}\right)^{\sigma} l^*(\kappa) \;\text{ and }\; l^{FE}(\kappa) = \left[(1-\theta)\left(\frac{1}{1+\iota}\right)^{\sigma} + \theta\right] l^*(\kappa). \qquad (45)$$

Comparing (44) to (45) shows that $l^{ZE}(\kappa) < l^{FE}(\kappa) < l^*(\kappa)$, i.e. for a given κ, hours worked are lower than optimal in both the FE and the ZE, and they are lower in the ZE than in the FE. The latter reflects that the friction created by the liquidity constraint on consumption is mitigated in the FE since part of households can pay with interest-bearing deposits.

Further, we have from (10) that consumption in the first best satisfies $c^*(\kappa) = [w(\kappa)]^{\sigma}$, while we have from (26)-(27) that consumption levels of households in state 0 and 1 (c_0 and c_1) in the FE and the ZE satisfy

$$c_0^{FE}(\kappa) = c_0^{ZE}(\kappa) = c_1^{ZE}(\kappa) = \left(\frac{1}{1+\iota}\right)^{\sigma} c^*(\kappa) \quad \text{and} \quad c_1^{FE}(\kappa) = c^*(\kappa). \qquad (46)$$

Expression (46) shows that, given κ, consumption levels are closer to the first best in the FE. The reason is again that the friction created by households' liquidity constraint is mitigated in the FE; in particular, providing liquidity to households paying with deposits entails no opportunity cost in the FE, and c_1^{FE} thus equals the first-best consumption level c^*.

The reason why a general welfare ranking of the FE and the ZE is not possible is that the inefficiency created by the cash-in-advance constraint for a fraction $1-\theta$ of households does not disappear in either equilibrium, as evidenced by the fact that hours worked $l(\kappa)$ are inefficiently low in both equilibria. From the theory of the second best, we know that correcting an inefficiently low $l(\kappa)$ with an inefficiently high κ can in principle increase welfare.[22] Note in particular that, since $\kappa^{FE} < \kappa^{ZE}$, we have $c_0^{FE} < c_0^{ZE}$ regardless of whether the FE or the ZE is the high-activity equilibrium, and c_0^{ZE} may be closer to c^* than c_0^{FE}. In other words, the overinvestment generated in the ZE may partly correct for the inefficiently low output resulting from the cash-in-advance friction; however, this comes at the cost of another distortion, namely that a given output is produced with an inefficiently high capital-labour ratio. Note also that, as the share of households paying with deposits (θ) converges to one, the FE converges to the first-best allocation.

It is instructive to relate these results to Geromichalos and Herrenbrueck (2022), who find in a similar model that the first best can be implemented by setting the

[22] The matter is further complicated by the fact that, as is well known, the first-best allocation does not maximize steady-state utility, which is maximized when the capital stock is at the golden rule level, $f'(\kappa) = \delta$. For this reason, a capital stock exceeding κ^* (as is the case in the ZE) may be associated with a higher steady-state utility than the first-best allocation.

interest rate i to the appropriate level within the bounds given by the zero-lower bound and the Fisher rate ι. In their model, like in ours, output, investment and consumption can be inefficiently low because household consumption is subject to a liquidity constraint. Similar to our model, lowering i raises investment, which spurs output and consumption and can thus correct for the inefficiency created by households' liquidity constraint. The key difference from our model is that in Geromichalos and Herrenbrueck (2022), there is only one relevant margin, which can be set to the first best by manipulating i. In particular, different to Geromichalos and Herrenbrueck (2022), we do not assume a fixed labour supply, which means that it is generally not possible to achieve the first best with an inefficiently large capital stock: while it may in principle be possible to bring output to its first-best level by reducing i below ι and thus creating some overinvestment, production will occur with an inefficient mix of capital and labour.

5.3 Transitional Dynamics

The fact that there can be multiple steady states does not yet imply that, starting from some initial capital stock (the model's only state variable), the economy can transition to different steady states. In Appendix A.3, we study the transitional dynamics of the model, and we confirm by example that the model can indeed exhibit multiple dynamic equilibria, i.e. multiple transition paths (each leading to a different steady state) are possible starting from some initial capital stock. However, we also provide an example of an economy that exhibits multiple steady states but a unique dynamic equilibrium starting from any initial capital stock. Note that the existence of multiple steady states can still be policy relevant if the dynamic equilibrium is unique, since this suggests that the economy may not transition back to the previous steady state after an (unexpected) shock to capital.

We will briefly describe here the solution method that we apply in Appendix A.3 to study the dynamics of the model and to examine whether multiple dynamic equilibria are possible. We refer to the appendix for further details on the method as well as on the numerical examples. Due to the highly non-linear nature of our model, we focus on global dynamics for parametrised examples. The solution technique we apply resembles a backward-shooting algorithm in the spirit of Judd (1999, Chapter 10.7) and Brunner and Strulik (2002). In short, the method works as follows. We first parametrise the model in such a way that all three types of steady state equilibrium (FE, ZE and IE) coexist, and we linearise the model around the steady states. In all the examples we consider, the FE and the ZE are locally saddle-path stable, while the IE is locally unstable. Next, we sample points on the saddle path in an ϵ-

neighbourhood around the FE and the ZE and, starting from these points, we move backwards in time, using the solution of the non-linear model. Importantly, even if the economy exhibits multiple equilibria, this backward iteration is unique. This method can demonstrate the existence of multiple dynamic equilibria if, starting from the two steady states and tracing the evolution of the economy backwards in time, the paths of the capital stock in the two solutions overlap. If this is the case, then this implies that—starting from certain initial values of the capital stock—the economy can move forward in ways that lead it to the saddle path towards one or the other steady state. Our examples in Appendix A.3 show that backward iterating from different steady states can, but does not have to, lead to overlapping paths for the capital stock. Thus, multiple dynamic equilibria are possible, but their existence cannot be taken as given when multiple steady states exist.

6 Conclusion

In this paper, we studied liquidity transformation by banks in a monetary model, and we analysed the channels through which liquidity transformation increases aggregate output and investment. We showed how liquidity transformation can lead to macroeconomic instability in the sense that it can lead to multiple steady-state equilibria with different interest rates in economies that do not exhibit such equilibrium multiplicity without liquidity transformation. The paper also makes a methodological contribution by showing how banks can be introduced in a cash-in-advance model. Key results from the New Monetarist macro-financial literature, e.g. on the emergence of liquidity traps, can be replicated within the CIA framework.

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Appendix A Additional Material

A.1 Nonlinear Disutility of Labour

Suppose that households' lifetime preferences are

$$U = \sum_{t=0}^{\infty} \beta^t \Big[u(c_t) - v(l_t) \Big], \tag{A.1}$$

where $v(l_t)$ is some continuously differentiable, increasing and strictly convex function. Following the procedure in Section 4, the bank's value function writes

$$V(m_{-1}, b_{-1}, k_{-1}) = \max_{c_0, c_1, l, m, b, k \geq 0} \theta u(c_1) + (1-\theta) u(c_0) - v(l) + \beta V(m, b, k). \tag{A.2}$$

The bank then solves (A.2) subject to the constraints (20)-(22). As in Section 4, we denote λ as the multiplier on the budget constraint (20), μ_0 as the multiplier on the liquidity constraint for households in state 0, (21), and μ_1 as the multiplier on the liquidity constraint for households in state 1, (22). Notice that the Fisher rate can be expressed as

$$1 + \iota \equiv \frac{1+\pi}{\beta} \frac{\lambda}{\lambda_{-1}}. \tag{A.3}$$

The first-order condition for l now writes

$$\lambda = \frac{v'(l)}{w}, \tag{A.4}$$

while the remaining first-order conditions are still given by (B.9)-(B.13). Combining (A.4), (B.11) and (B.12), we obtain

$$\mu_0 = \frac{v'(l)}{w} \frac{i(1+\iota)}{1+i} \quad \text{and} \quad \mu_1 = \frac{v'(l)}{w} \left(\frac{\iota - i}{1+i} \right). \tag{A.5}$$

Combining these with the other first-order conditions ((B.9), (B.10) and (B.13)), we obtain

$$u'(c_0) = \frac{v'(l)}{w}(1+\iota), \tag{A.6}$$

$$u'(c_1) = \frac{v'(l)}{w} \frac{1+\iota}{1+i}, \tag{A.7}$$

$$\psi + 1 - \delta = \frac{1}{\beta} \frac{w}{w_{-1}} \frac{v'(l_{-1})}{v'(l)} \frac{1+i}{1+i+\chi(\iota - i)}. \tag{A.8}$$

We now impose steady state for the remainder of this subsection. Recall first from expression (31) that we can use the aggregate resource constraint to express hours worked in a steady state as a function of c_0, c_1 and κ: $L = l = l(c_0, c_1, \kappa)$. Note that we have $\partial l / \partial c_0 > 0$, $\partial l / \partial c_1 > 0$, and $\partial l / \partial \kappa > 0 \, (< 0)$ if $f'(\kappa) > \delta \, (f'(\kappa) < \delta)$. Then, evaluating (A.6)-(A.8) in steady state, and using $\psi = f'(\kappa)$ and $w(\kappa) = f(\kappa) - f'(\kappa)\kappa$, we obtain

$$u'(c_0) = \frac{v'(l(c_0, c_1, \kappa))}{w(\kappa)}(1+\iota), \tag{A.9}$$

$$u'(c_1) = \frac{v'(l(c_0, c_1, \kappa))}{w(\kappa)} \frac{1 + \iota}{1 + i}, \tag{A.10}$$

$$f'(\kappa) + 1 - \delta = \frac{1}{\beta} \frac{1 + i}{1 + i + \chi(\iota - i)}. \tag{A.11}$$

Conditions (A.9)-(A.11) jointly pin down (c_0, c_1, κ) given i, while the equilibrium interest rate i is determined by the asset market clearing condition (36). Thus, a steady-state equilibrium in the economy with nonlinear disutility of labour is characterised by a vector (c_0, c_1, κ, i) solving (36) and (A.9)-(A.11) with $1 + \iota = (1 + \gamma)/\beta$.

The only difference from the steady-state equilibrium in the quasilinear economy discussed in Sections 4-5 is that hours worked l now appear in the first-order conditions for consumption levels c_0 and c_1. This introduces some additional general equilibrium effects that are not present in the quasilinear economy.

We can discuss informally how the assumption of nonlinear labour disutility will affect the results derived for the quasilinear economy, specifically with regard to the relationship between the interest rate i, consumption and output. The channels through which i affects consumption in the quasilinear economy are evidently still present in the nonlinear economy: changes in i affect optimal consumption levels by changing the real wage and the opportunity cost of providing liquidity to households in state 1. However, there are now additional channels. First, the fact that κ falls when i increases means that working hours needed to produce a given output increase. This negatively affects optimal consumption levels by making leisure relatively more attractive, which reinforces the effect coming from the fall in the real wage. An important qualification is that this only holds if $f'(\kappa) \geq \delta$, i.e. the economy is not starting from a situation of inefficient overinvestment. If the economy is characterised by inefficient overinvestment—which may well be the case in a ZE—then a fall in κ will *reduce* the hours needed to produce a given output and the effect reverses. Furthermore, changes in aggregate consumption (which imply a change in hours worked, holding κ fixed) now have an independent effect on optimal consumption levels c_0 and c_1. Thus, different from the quasilinear economy, knowing i (and hence κ) is not sufficient to derive the optimal consumption of households in a given state—rather, the optimal consumption levels of households in both states are determined jointly.

These additional general equilibrium effects associated with the assumption of nonlinear disutility of labour complicate the analysis of the model, and they are arguably not central to the main mechanisms that we want to highlight in this paper. For this reason, we focus on the case with linear labour disutility to present the key insights from our setup in the most transparent way.

A.2 Example of Multiple Equilibria with a Constant Money-to-Debt Ratio

Figure 3 shows an example of multiple equilibria in an economy with a constant money-to-debt ratio of $\eta = 0.45$. The top left panel shows aggregate demand and supply for indirectly-liquid assets, $\mathcal{A}^d(i)$ and $\mathcal{A}^s(i)$. From the graph, it is clear that the conditions for all three types of equilibria from (36) are satisfied simultaneously, with the IE existing for an interest rate of 3.4%. We can also see that the FE is the high-activity equilibrium in this example. Notice that $K(i)$ is increasing over most of i while the aggregate supply of indirectly liquid asset, $\mathcal{A}^s(i)$, falls in i. The reason for this is that, as described in the main text, a higher interest rate i implies a lower real aggregate bond supply when the money-to-bonds ratio is fixed; as a result, $K(i)$ and $\mathcal{A}^s(i)$ can move in opposite directions when i changes.

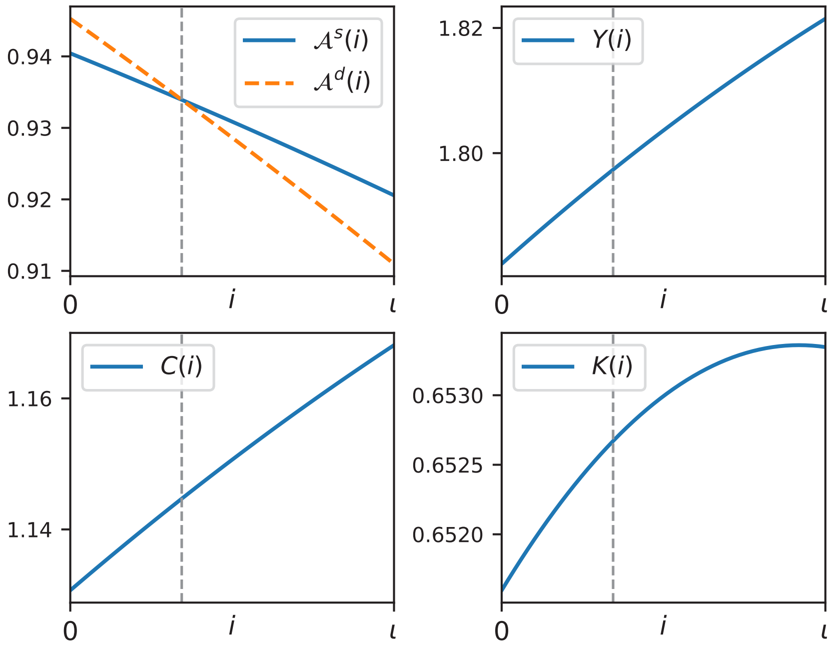


Figure 3: Example of multiple equilibria with a constant money-to-debt ratio. The parameters for this example are: $\beta = 0.95, \gamma = 0.045, \theta = 0.8, \sigma = 0.7, D = 1, A = 2, \alpha = 0.1, \rho = 1/5, \chi = 1, \delta = 1, \eta = 0.45$. The Fisher rate equals $\iota = 0.1$.

A.3 Transitional Dynamics

The existence of multiple steady states in itself does not imply that, starting from some initial state, the economy can transition to different steady-state equilibria. Since the amount of capital K_{-1} is the only state variable in this economy, demonstrating the existence of multiple dynamic equilibria requires us to show that the economy can transition to different steady states starting from the same initial capital stock. In this section, we investigate with numerical examples the transitional dynamics of the model to determine whether the model can exhibit multiple dynamic equilibria. We focus on the case with a fixed real supply of government bonds, i.e. $\mathcal{B}(\cdot) = \overline{\mathcal{B}}$.

The evolution of the real side of the economy can be described by the evolution of the vector $X = (\kappa, K, Z)$, where $Z = \phi M_{-1}$ denotes real money balances. The solution technique we apply relies on backward iteration, and in particular on the following proposition, the proof of which is given in Appendix B.9:

Proposition A.1. *If $\mathcal{B}(\cdot) = \overline{\mathcal{B}}$ and $-cu''(c)/u'(c) < 1$ for all c, then, given $X = (\kappa, K, Z)$, there is a unique X_{-1} such that the equilibrium conditions hold.*

Proposition A.1 states that there is a unique mapping from $X \mapsto X_{-1}$, i.e. starting from some date t and some value of X, the model's solution implies a unique path of X backwards in time.[23] We refer to this map as $g : X \mapsto X_{-1}$ in what follows, and we use it to backward iterate from some terminal X_T. We thus adopt a backward-shooting approach in the spirit of Judd (1999, Chapter 10.7) and Brunner and Strulik (2002).[24]

Next, we denote $\bar{X}_{FE}$ and $\bar{X}_{ZE}$ as the values of X that describe the FE and ZE steady states, respectively. Furthermore, we define

$$\mathscr{B}_\epsilon(\bar{X}) \equiv \left\{ X \quad \text{s.t.} \quad \left| \frac{X - \bar{X}}{\bar{X}} \right| \leq \epsilon \right\}, \tag{A.12}$$

as an ϵ-neighbourhood around $\bar{X}$, where $|\cdot|$ denotes the Euclidean norm. Note that we cannot start the backward iteration at the steady-state values since $g(\bar{X}_{FE}) = \bar{X}_{FE}$ and $g(\bar{X}_{ZE}) = \bar{X}_{ZE}$. Instead, we linearise the model around the steady states to derive sets $\mathscr{E}_{\epsilon,FE} \subseteq \mathscr{B}_\epsilon(\bar{X}_{FE})$ and $\mathscr{E}_{\epsilon,ZE} \subseteq \mathscr{B}_\epsilon(\bar{X}_{ZE})$ such that if $X \in \mathscr{E}_{\epsilon,FE}$, the linearised economy will transition to $\bar{X}_{FE}$; and if $X \in \mathscr{E}_{\epsilon,ZE}$, the linearised economy will transition to $\bar{X}_{ZE}$.[25] In other words, $\mathscr{E}_{\varepsilon,FE}$ and $\mathscr{E}_{\varepsilon,ZE}$ are the stable manifolds of the FE and the ZE, respectively, in the linearised model.

Slightly abusing notation, let $g(\mathscr{X})$ be the image of $\mathscr{X}$ through g for some subset $\mathscr{X}$ of the domain of g. Furthermore, let $g_K(X)$ refer to the value of K in the vector $g(X)$. Given this, we can define

$$\mathscr{K}_{\varepsilon,J} \equiv \bigcup_{T=0}^{\infty} g_K^T(\mathscr{E}_{\varepsilon,J}), \quad J \in \{FE, ZE\}, \tag{A.13}$$

where $\mathscr{K}_{\varepsilon,J}$ is the set of values of K that the economy passes through when moving backwards in time from the stable manifold of steady state $J \in \{FE, ZE\}$. Thus, if

[23] Importantly, this does not imply that that the inverse is true, i.e. that $X_{-1} \mapsto X$. In fact, multiple dynamic equilibria only exist if this is not the case.

[24] Judd (1999, Chapter 10.7) and Brunner and Strulik (2002) discuss the fact that standard forward-shooting algorithms have limited value in models with saddle-path equilibria since these algorithms are extremely sensitive to small errors in the initial guess.

[25] We refer to Appendix A.3.1 for the details on the linearisation.

$K_{-1} \in \mathscr{K}_{\varepsilon,J}$, then the approximation suggests that for starting value K_{-1}, there exists an equilibrium path that converges to steady state $J \in \{FE, ZE\}$.

For all parametrisations we consider, we find that both the FE and ZE equilibrium are locally saddle-path stable, meaning that $\mathscr{E}_{\epsilon,FE}$ and $\mathscr{E}_{\epsilon,ZE}$ are straight lines. Formally, we have

$$\mathscr{E}_{\varepsilon,J} = \{\bar{X}_J \odot (1 + \lambda \hat{X}_J), \quad \forall \lambda \in [-\epsilon, \epsilon]\}, \quad J \in \{FE, ZE\}, \tag{A.14}$$

where $\odot$ denotes the Hadamard product, and where $\hat{X}_J = [\hat{\kappa}_J, \hat{K}_J, \hat{Z}_J]$ is uniquely determined up to some normalisation—we set its Euclidean length to one. Then, for each steady state $J \in \{FE, ZE\}$, we take a discrete sample of N points, $\{e_{n,J}\}_{n=0}^N$, on the saddle path, and we iterate backwards in time from these points.[26] Although this procedure yields 'only' single points in $g^T(\mathscr{E}_{\epsilon,J})$, these points allow us to say more about the entire set $g^T(\mathscr{E}_{\epsilon,J})$. The reason for this is that the proof of Proposition A.1 implies:

Corollary 1. *The map $g(X)$ is continuous in X.*

The implication of this is that the image $g(\mathscr{E}_{\epsilon,J})$ is compact and connected, since $\mathscr{E}_{\epsilon,J}$, as defined in (A.14), is compact and connected as well. In other words, if $K'_{-1}, K''_{-1} \in g_K^T(\mathscr{E}_{\epsilon,J})$, with $K'_{-1} < K''_{-1}$ then $[K'_{-1}, K''_{-1}] \subseteq g^T(\mathscr{E}_{\epsilon,J})$. This means that for any value of K_{-1} that lies between the most extreme values obtained from our sampling procedure, there is a transition path to some point on the saddle path towards the steady state that we backward-iterate from. Thus, given the sample $\{e_{n,J}\}_{n=0}^N \in \mathscr{E}_{\epsilon,J}$, our best approximation of $g_K^T(\mathscr{E}_{\epsilon,J})$ is

$$\tilde{g}_K^T(\{e_{n,J}\}_{n=0}^N) \equiv \left[\min_{e \in \{e_{n,J}\}_{n=0}^N} \{g_K^T(e)\}, \max_{e \in \{e_{n,J}\}_{n=0}^N} \{g_K^T(e)\} \right]. \tag{A.15}$$

We now apply this procedure to two parametrised examples. We start by considering the parametrisation described in Table 1, with corresponding steady-state values given in Table 2. Note that for this parametrisation, an FE, an IE and a ZE all exist. Figure 4 plots the results from our backwards iteration procedure. We see that the values for K_{-1} that we obtain by backwards iterating from the FE and ZE steady states do not overlap; instead, backwards iterating from the FE steady state leads K_{-1} to approach the IE steady state from below, while backwards iterating from the ZE steady state approaches the IE steady state from above. This shows several things: first, the IE steady state is locally unstable, which we confirm numerically by analysing the linearised model; second, for $K \in (0, \bar{K}_{IE})$, we transition to the FE steady state, while for $K \in (\bar{K}_{IE}, \infty)$, we transition to the ZE steady state; finally, this implies that although this parametrisation allows for multiple steady states, the dynamic equilibrium starting from any initial capital

[26] Specifically, we sample $e_{n,J} = \bar{X}_J \odot (1 + \lambda_n \hat{X}_J)$ with $\lambda_n = \{-100, -99, ..., -1, 1, 2, ..., 100\} \times e^{-9}$.

Table 1: Parametrisation where the dynamic equilibrium is unique.

β	γ	θ	σ	D	A	α	ρ	χ	δ	$\mathcal{B}$
0.98	0.02	0.8	0.7	1.0	2.0	0.1	0.3333	1.0	0.6	0.6250

Table 2: Steady state values for parametrisation where the dynamic equilibrium is unique.

	κ	K	Z	C	Y	L	π	ι	i
FE steady state	0.6413	0.7420	0.3284	1.7192	2.1644	1.1571	2.00	4.08	4.08
IE steady state	0.6542	0.7457	0.3343	1.6937	2.1412	1.1399	2.00	4.08	1.16
ZE steady state	0.6596	0.7473	0.3381	1.6835	2.1319	1.1331	2.00	4.08	0.00

stock is unique.

Next, consider the parametrisation characterised in Table 3, with corresponding steady-state values in Table 4. The results from the backward iteration for this parametrisation are presented in Figure 5. In this case, we observe a clear overlap for values of K_{-1}, which means that for certain values of the initial capital stock, the economy may either transition to the FE or the ZE steady state. Note further that as T increases, both paths again approach the IE steady state, implying that the IE equilibrium is again unstable, which we again confirm numerically with the linearised model.

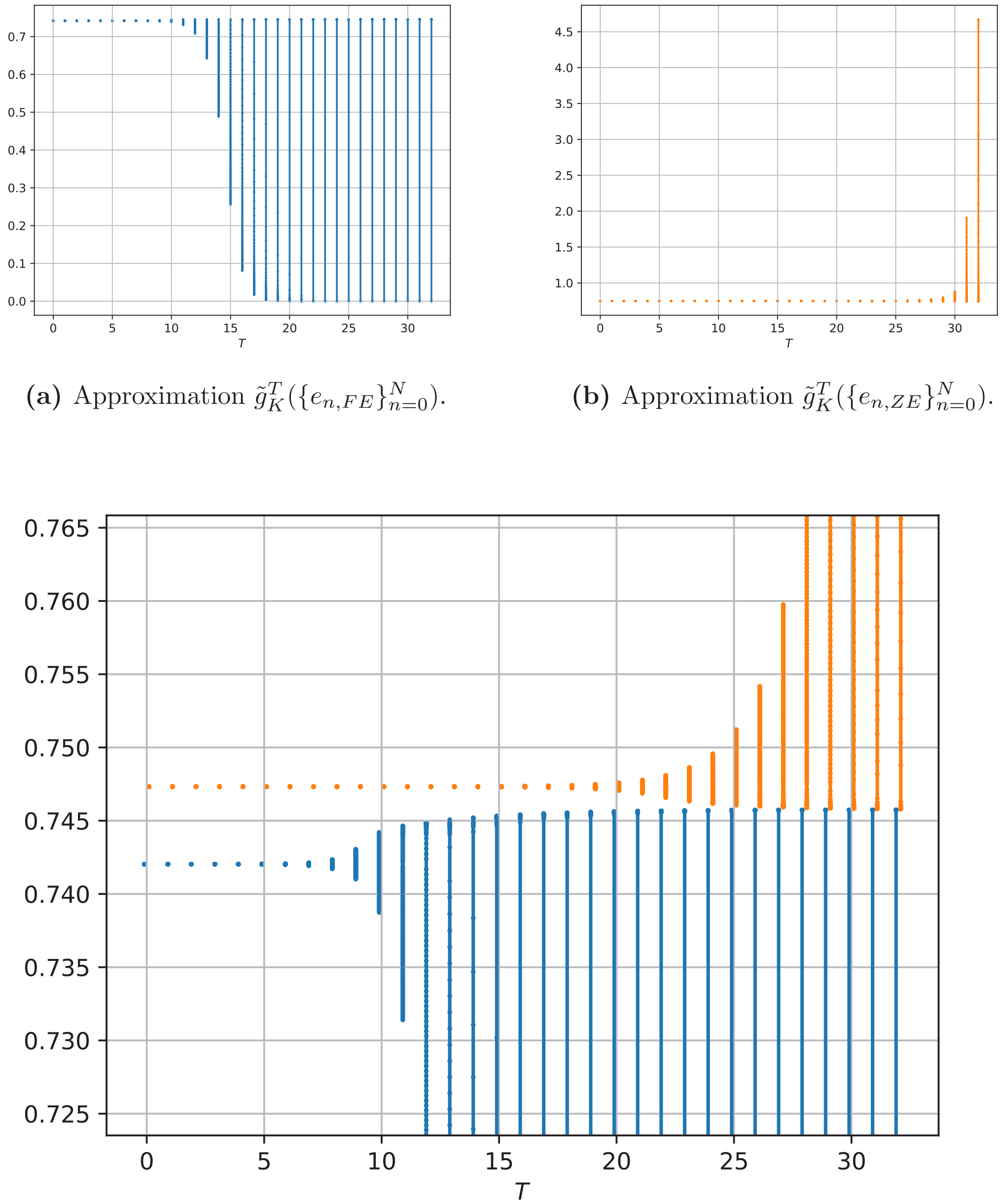
In Figure 6, we plot two equilibrium trajectories starting from $K_{-1} = 0.5522$: one that moves to the FE steady state and one that moves to the ZE steady state. To do so, we first calculate

$$\hat{e}_J = \underset{e \in \{e_{n,J}\}_{n=0}^N}{\arg\min} \; |g_K^T(e) - K_{-1}|, \quad J \in \{FE, ZE\}, \tag{A.16}$$

i.e. among the sampled trajectories moving towards the saddle path around steady state $J \in \{FE, ZE\}$, we take the one that starts out closest to $K_{-1} = 0.5522$. We set $T = 30$ because Figure 5 suggests that $K_{-1} = 0.5522 \in g_K^T(\mathscr{E}_{\epsilon,FE}) \cap g_K^T(\mathscr{E}_{\epsilon,ZE})$ for $T = 30$. Then, we use as approximate equilibrium trajectories $\{X_{t,J}\}_{t=0}^T = \{g^{T-t}(\hat{e}_J)\}_{t=0}^T$ for $J \in \{FE, ZE\}$. One interpretation is that after a shock to capital such that $K_{-1} = 0.5522$, the economy coordinates on a path that will lead it either to an FE or to a ZE steady state, and that output, interest rates, and inflation adjust accordingly. As the figure shows, the real implications of the two equilibria are markedly different, both on the transition path as well as in the corresponding steady state.

Table 3: Parametrisation with multiple dynamic equilibria.

β	γ	θ	σ	D	A	α	ρ	χ	δ	$\mathcal{B}$
0.98	0.02	0.8	0.7	1.0	2.0	0.1	0.3333	1.0	1.0	0.5225



(a) Approximation $\tilde{g}_K^T(\{e_{n,FE}\}_{n=0}^N)$.

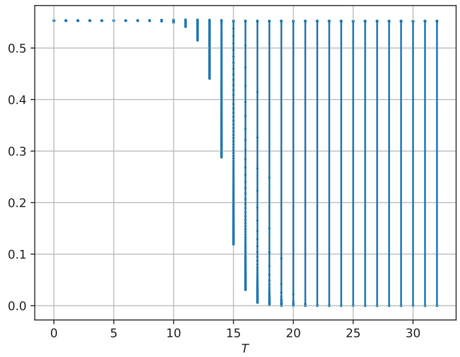
(b) Approximation $\tilde{g}_K^T(\{e_{n,ZE}\}_{n=0}^N)$.

(c) Approximations $\tilde{g}_K^T(\{e_{n,FE}\}_{n=0}^N)$ (●) and $\tilde{g}_K^T(\{e_{n,ZE}\}_{n=0}^N)$ (●). The range of K is capped from below and from above for enhanced visibility.

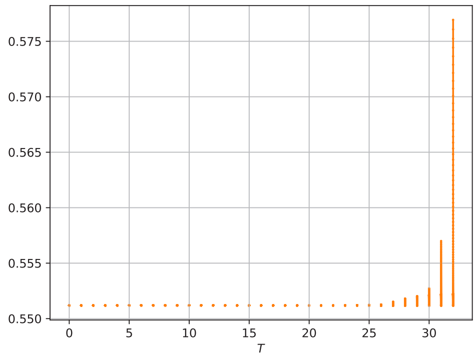
Figure 4: Approximations $\tilde{g}_K^T(\{e_{n,FE}\}_{n=0}^N)$ and $\tilde{g}_K^T(\{e_{n,ZE}\}_{n=0}^N)$ for parametrisation where the dynamic equilibrium is unique.

Table 4: Steady state values for parametrisation with multiple dynamic equilibria.

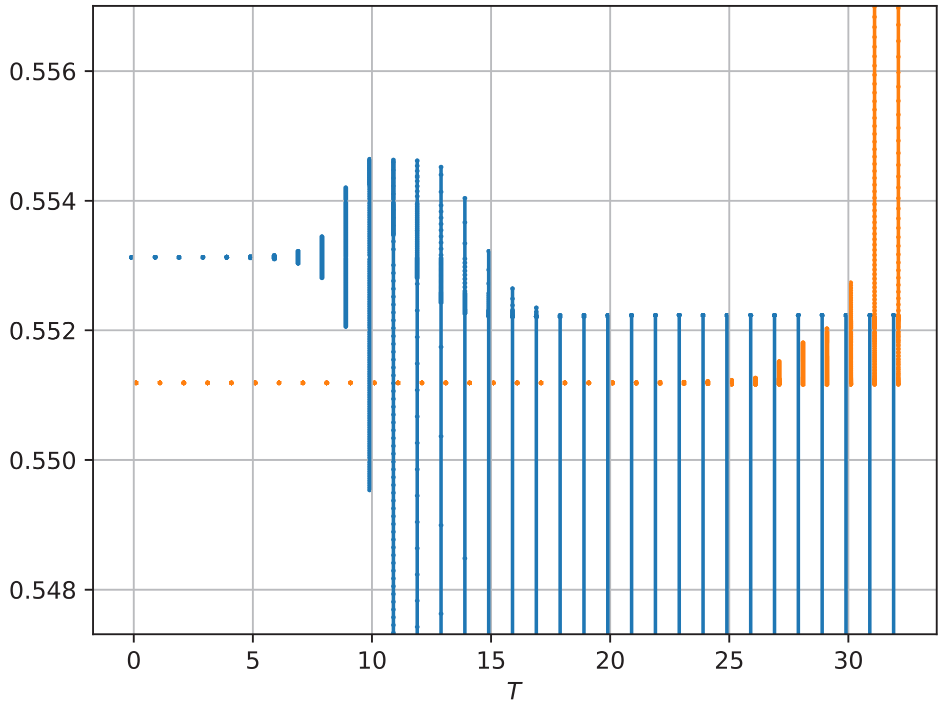
	κ	K	Z	C	Y	L	π	ι	i
FE steady state	0.5136	0.5531	0.2581	1.3513	1.9045	1.0769	2.00	4.08	4.08
IE steady state	0.5182	0.5522	0.2610	1.3372	1.8895	1.0657	2.00	4.08	2.14
ZE steady state	0.5234	0.5512	0.2688	1.3215	1.8726	1.0531	2.00	4.08	0.00



(a) Approximation $\tilde{g}_K^T(\{e_{n,FE}\}_{n=0}^N)$.

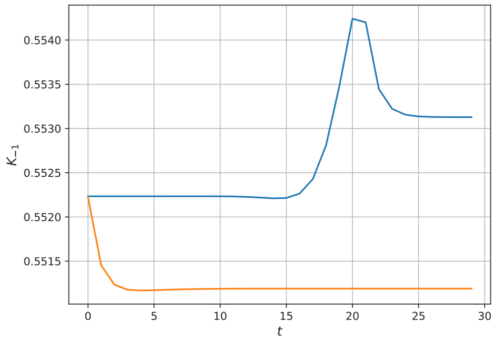


(b) Approximation $\tilde{g}_K^T(\{e_{n,ZE}\}_{n=0}^N)$.

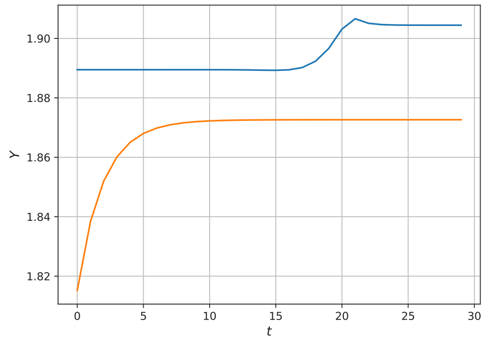


(c) Approximations $\tilde{g}_K^T(\{e_{n,FE}\}_{n=0}^N)$ (●) and $\tilde{g}_K^T(\{e_{n,ZE}\}_{n=0}^N)$ (●). The range of K is capped from below and from above for enhanced visibility.

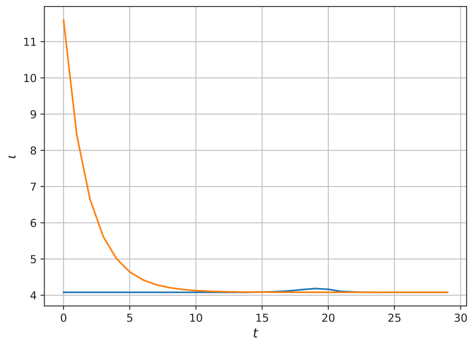
Figure 5: Approximations $\tilde{g}_K^T(\{e_{n,FE}\}_{n=0}^N)$ and $\tilde{g}_K^T(\{e_{n,ZE}\}_{n=0}^N)$ for parametrisation with multiple dynamic equilibria.



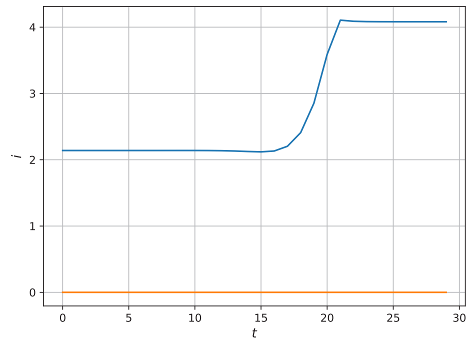
(a) Dynamics for K_{t-1}



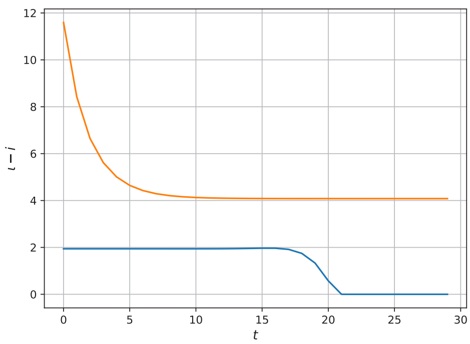
(b) Dynamics for Y_t



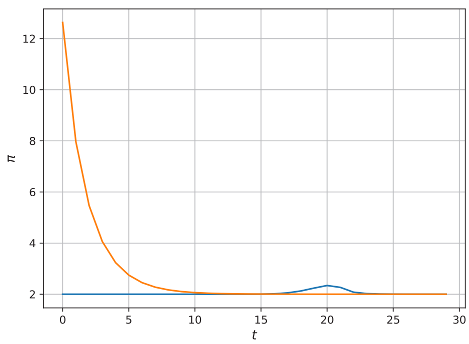
(c) Dynamics for ι_t



(d) Dynamics for i_t



(e) Dynamics for the spread $\iota_t - i_t$



(f) Dynamics for π_t

Figure 6: Equilibrium dynamics with $K_{-1} = 0.5522$ for parametrisation which allows for multiple dynamic equilibria. There is an equilibrium path to the FE (●) steady state and an equilibrium path to the ZE (●) steady state.

A.3.1 Details on the Linearisation

Given the map $g : X \to X_{-1}$, we can log-linearise around a steady state $\bar{X}$ to obtain

$$\hat{X}_{-1} = \bar{\varepsilon}_{g,\bar{X}} \hat{X}, \tag{A.17}$$

where $\hat{X} \equiv (X - \bar{X}) \oslash X$, $\bar{\varepsilon}_{g,X}$ is a 3-by-3 matrix, and $\oslash$ denotes the Hadamard division.

To find how $\bar{\varepsilon}_{g,X}$ looks like for either an FE, ZE, or IE steady state, it is useful to first log-linearise equations (B.34), (B.37) and (B.39). Denoting deviations from the steady state with a hat, this yields

$$\hat{\kappa}_{-1} = \frac{\bar{\varepsilon}_{w,\kappa} - \bar{\varepsilon}_{\tilde{\psi},\kappa}}{\bar{\varepsilon}_{w,\kappa}} \hat{\kappa} + \frac{1}{\bar{\varepsilon}_{w,\kappa}} \frac{\chi(1+\bar{\iota})}{(1-\chi)(1+\bar{\iota}) + \chi(1+\bar{i})} \left(\widehat{1+i} - \widehat{1+\iota} \right), \tag{A.18}$$

$$\hat{K}_{-1} = \frac{1}{\sigma} \frac{1}{\tilde{\varphi}(\bar{\kappa})\bar{K}} \left[\bar{C}(\bar{\varepsilon}_{w,\kappa}\hat{\kappa} - \widehat{1+\iota}) + \theta \bar{c}_1 \widehat{1+i} \right] + \frac{\hat{K}}{\tilde{\varphi}(\bar{\kappa})} - \bar{\varepsilon}_{\tilde{\varphi},\kappa}\hat{\kappa}, \tag{A.19}$$

$$\hat{Z}_{-1} = \frac{(1-\chi)(1+\bar{i})}{(1-\chi)(1+\bar{i}) + \chi(1+\bar{\iota})} \widehat{1+\iota} + \frac{\chi(1+\bar{\iota})}{(1-\chi)(1+\bar{i}) + \chi(1+\bar{\iota})} \widehat{1+i} + \hat{Z} - \bar{\varepsilon}_{\tilde{\psi},\kappa}\hat{\kappa}, \tag{A.20}$$

where we defined

$$\bar{\varepsilon}_{\tilde{\psi},\kappa} \equiv \frac{\bar{\kappa} f''(\bar{\kappa})}{f'(\bar{\kappa}) + 1 - \delta}, \quad \bar{\varepsilon}_{\tilde{\varphi},\kappa} \equiv \frac{f'(\bar{\kappa}) - f(\bar{\kappa})/\bar{\kappa}}{f(\bar{\kappa})/\bar{\kappa} + 1 - \delta} \quad \text{and} \quad \bar{\varepsilon}_{w,\kappa} \equiv - \frac{\bar{\kappa}^2 f''(\bar{\kappa})}{f(\bar{\kappa}) - \bar{\kappa} f'(\bar{\kappa})}. \tag{A.21}$$

As seen above, market clearing for money and indirectly-liquid assets pins down (ι, i) as a function of X. Hence, we can think of the log-linearised equations for money and asset market clearance as

$$\widehat{1+\iota} = \bar{\varepsilon}_{\iota,\kappa}\hat{\kappa} + \bar{\varepsilon}_{\iota,K}\hat{K} + \bar{\varepsilon}_{\iota,Z}\hat{Z}, \tag{A.22}$$

$$\widehat{1+i} = \bar{\varepsilon}_{i,\kappa}\hat{\kappa} + \bar{\varepsilon}_{i,K}\hat{K} + \bar{\varepsilon}_{i,Z}\hat{Z}, \tag{A.23}$$

where $\bar{\varepsilon}_{\iota,\kappa}$, $\bar{\varepsilon}_{\iota,K}$, $\bar{\varepsilon}_{\iota,Z}$, $\bar{\varepsilon}_{i,\kappa}$, $\bar{\varepsilon}_{i,K}$, and $\bar{\varepsilon}_{i,Z}$ follow from log-linearisation as shown below. Using equations (A.22) and (A.23) in the system (A.18)-(A.20) then implicitly gives $\bar{\varepsilon}_{g,X}$.

FE steady state. In an FE, we have $\bar{i} = \bar{\iota}$, such that $\widehat{1+i} = \widehat{1+\iota}$. Since the money market clearance condition (B.41) must hold with equality in an FE, we can log-linearise it to obtain

$$\widehat{1+\iota} = \bar{\varepsilon}_{w,\kappa}\hat{\kappa} - \sigma\hat{Z}. \tag{A.24}$$

ZE steady state. In a ZE, we have $\bar{i} = 0$, such that $\widehat{1+i} = 0$. The asset market clearance condition (B.46) pins down $1+\iota$, and it can be log-linearised as

$$\begin{aligned} &\left[\frac{1}{\sigma} - \frac{\varpi_{\mathcal{B}}(1-\chi)(1+\bar{i})}{(1-\chi)(1+\bar{i}) + \chi(1+\bar{\iota})} \right] \widehat{1+\iota} \\ &= \left[\frac{\bar{\varepsilon}_{w,\kappa}}{\sigma} + \left(\varpi_K + \frac{\chi\tilde{\psi}(\bar{\kappa})}{\tilde{\varphi}(\bar{\kappa}) - \chi\tilde{\psi}(\bar{\kappa})} \right) (\bar{\varepsilon}_{\varphi,\kappa} - \bar{\varepsilon}_{\psi,\kappa}) - \varpi_{\mathcal{B}}\bar{\varepsilon}_{\psi,\kappa} \right] \hat{\kappa} - \varpi_K\hat{K} - \varpi_Z\hat{Z}, \end{aligned} \tag{A.25}$$

where

$$\varpi_Z = \frac{Z}{C}\frac{\tilde{\varphi}(\bar{\kappa})}{\tilde{\varphi}(\bar{\kappa}) - \chi\tilde{\psi}(\bar{\kappa})}, \quad \varpi_K = \frac{K}{C}\frac{\chi\tilde{\psi}(\bar{\kappa})}{\tilde{\varphi}(\bar{\kappa}) - \chi\tilde{\psi}(\bar{\kappa})}$$
$$\text{and} \quad \varpi_{\mathcal{B}} = \frac{\tilde{\psi}(\bar{\kappa})\overline{\mathcal{B}}}{C}\frac{\tilde{\varphi}(\bar{\kappa})}{\tilde{\varphi}(\bar{\kappa}) - \chi\tilde{\psi}(\bar{\kappa})}\frac{(1-\chi)(1+\bar{i}) + \chi(1+\bar{\iota})}{1+\bar{\iota}}. \tag{A.26}$$

IE steady state. In an IE, we have $0 < \bar{i} < \bar{\iota}$, and both (B.41) and (B.46) must hold with equality. This gives the log-linearised equations (A.24) and

$$\begin{aligned}&\left[\frac{\theta}{\sigma}\frac{\bar{c}_1}{\bar{C}} - \frac{\varpi_{\mathcal{B}}(1-\chi)(1+\bar{i})}{(1-\chi)(1+\bar{i}) + \chi(1+\bar{\iota})}\right]\widehat{1+i} \\ &\qquad = -\left[\frac{\bar{\varepsilon}_{w,\kappa}}{\sigma} + \left(\varpi_K + \frac{\chi\tilde{\psi}(\bar{\kappa})}{\tilde{\varphi}(\bar{\kappa}) - \chi\tilde{\psi}(\bar{\kappa})}\right)(\bar{\varepsilon}_{\varphi,\kappa} - \bar{\varepsilon}_{\psi,\kappa}) - \varpi_{\mathcal{B}}\bar{\varepsilon}_{\psi,\kappa}\right]\hat{\kappa} \\ &\qquad\quad + \varpi_K\hat{K} + \varpi_Z\hat{Z} + \left(\frac{1}{\sigma} - \varpi_{\mathcal{B}}\frac{(1-\chi)(1+\bar{i})}{(1-\chi)(1+\bar{i}) + \chi(1+\bar{\iota})}\right)\widehat{1+\iota},\end{aligned} \tag{A.27}$$

where ϖ_Z, $\varpi_{\mathcal{B}}$, and ϖ_K are as in equation (A.26).

Appendix B Proofs

B.1 Proof of Lemma 1

We denote the Lagrange multipliers on the budget constraint and the liquidity constraint by λ and μ, respectively. We can ignore the non-negativity constraints on the household's choice variables, as they never bind in equilibrium: if either m or k were zero, consumption would necessarily be zero as well, but this is at odds with the Inada conditions. For bonds, demand cannot be negative in equilibrium, as supply is positive, and prices adjust such that the market clears. We then obtain the following first-order conditions for the household's problem:

$$l: \quad \lambda = \frac{1}{w} \tag{B.1}$$
$$c: \quad u'(c) = \lambda + \mu \tag{B.2}$$
$$m: \quad \phi\lambda = \beta\phi_{+1}[\lambda_{+1} + \mu_{+1}] \tag{B.3}$$
$$b: \quad \phi\lambda = \beta\phi_{+1}(1+i_{+1})\lambda_{+1} \tag{B.4}$$
$$k: \quad \lambda = \beta\big[\psi_{+1} + (1-\delta)\big]\lambda_{+1} \tag{B.5}$$

From (B.4) and (B.5) we obtain

$$\frac{1+i}{1+\pi} = \psi + 1 - \delta, \tag{B.6}$$

which is a no-arbitrage condition stating that bonds and capital need to earn the same real return. Combining (B.1) with (B.4), we obtain

$$\frac{1+i}{1+\pi} = \frac{1}{\beta}\frac{w}{w_{-1}}, \tag{B.7}$$

which together with (B.6) yields (13). Next, combining (B.2) with (B.3), we obtain $u'(c) = [(1+\pi)\lambda_{-1}]/\beta$. Substituting for λ_{-1} using (B.1) yields $u'(c) = (1+\pi)/(\beta w_{-1})$, and substituting for w_{-1} using (B.7) then leads to condition (14). Finally, from (B.3) and (B.4), we have $\mu = i\lambda$. It is straightforward that the household's budget constraint binds in equilibrium, i.e. we have $\lambda > 0$. Thus, when $i > 0$, we have $\mu > 0$, meaning that the CIA constraint binds. ■

B.2 Proof of Lemma 2

Denote λ as the Lagrange multiplier for the budget constraint and μ_Θ as the multipliers for the liquidity constraints of households in state $\Theta \in \{0, 1\}$. Using our definition of the Fisher rate from (15), the first-order conditions of the bank's problem are as follows:

$$l: \quad \lambda = \frac{1}{w} \tag{B.8}$$

$$c_0: \quad u'(c_0) = \lambda + \mu_0 + \mu_1 \tag{B.9}$$

$$c_1: \quad u'(c_1) = \lambda + \mu_1 \tag{B.10}$$

$$m: \quad \iota = \frac{\mu_0 + \mu_1}{\lambda} \tag{B.11}$$

$$b: \quad 1 + \iota = (1+i)\left(1 + \frac{\mu_1}{\lambda}\right) \tag{B.12}$$

$$k: \quad 1 + \iota = (1+\pi)(\psi + 1 - \delta)\left(1 + \frac{\chi\mu_1}{\lambda}\right) \tag{B.13}$$

From (B.11) and (B.12), we find that

$$\mu_0 = \frac{1}{w}\left[\frac{i(1+\iota)}{1+i}\right] \quad \text{and} \quad \mu_1 = \frac{1}{w}\left(\frac{\iota - i}{1+i}\right), \tag{B.14}$$

which shows that the liquidity constraint for households in state 0 binds whenever $i > 0$, while the liquidity constraint for households in state 1 binds whenever $i < \iota$.

Next, combining (B.14) with (B.9) and (B.10), and substituting $\lambda = 1/w$ from (B.8), we obtain

$$u'(c_0) = \frac{1+\iota}{w} \quad \text{and} \quad u'(c_1) = \frac{1}{w}\frac{1+\iota}{1+i}.$$

Finally, by combining (B.13) with our expression for μ_1 from (B.14), and substituting $1 + \pi = [(1+\iota)\beta w_{-1}]/w$ using the definition of the Fisher rate (15), we obtain

$$\psi + 1 - \delta = \frac{1}{\beta}\frac{w}{w_{-1}}\frac{1+i}{1+i+\chi(\iota - i)}.$$

■

B.3 Proof of Proposition 1

The equilibrium nominal rate i is pinned down by (16), and since $f(\kappa)$ is strictly concave, condition (17) uniquely determines κ. Since $u(c)$ is strictly concave, condition (18) then uniquely determines c given i and κ. ■

B.4 Proof of Lemma 3

Note first that if $Q(0) > 0$ and $Q'(i) < 0$ for $Q(i) \in [0, \iota]$, then there exists some constant $\overline{\mathcal{B}} > 0$ with $\overline{\mathcal{B}} \in (Q(\iota), Q(0))$. For any such $\overline{\mathcal{B}}$, we have $\overline{\mathcal{B}} > Q(\iota)$ and $\overline{\mathcal{B}} < Q(0)$ such that, by (38), both an FE and a ZE exist. Furthermore, since $Q(i)$ is continuous, we know from the intermediate value theorem that there exists some $i \in (0, \iota)$ such that $\overline{\mathcal{B}} = Q(i)$, which means that an IE exists as well. ■

B.5 Proof of Lemma 4

Denote

$$\mathcal{B}_\eta(i) = \frac{1-\eta}{\eta}\mathcal{M}(i) \tag{B.15}$$

as the real bond supply under a fixed money-to-debt ratio $\eta \in (0, 1)$.

Step 1: $i_H \geq i_L \Rightarrow c_0(i_H) \leq c_0(i_L)$. This follows from the fact that, by (26), $c_0(i)$ is strictly increasing in w, which itself is strictly increasing in κ, which, in turn, is weakly decreasing in i (see (28)).

Step 2: $i_H \geq i_L \Rightarrow \mathcal{B}_\eta(i_H) \leq \mathcal{B}_\eta(i_L)$. This follows immediately from (30) and (B.15) together with the result in Step 1.

Suppose an FE and a ZE coexist for some fixed money-to-debt ratio η. According to (33) and (36), this implies that

$$\mathcal{B}_\eta(\iota) + \chi K(\iota) \geq \mathcal{A}^d(\iota) \quad \text{and} \quad \mathcal{B}_\eta(0) + \frac{\chi(1+\iota)}{1+\chi\iota}K(0) \leq \mathcal{A}^d(0). \tag{B.16}$$

From Step 2, we know that there exists a strictly positive constant $\overline{\mathcal{B}} \in [\mathcal{B}_\eta(\iota), \mathcal{B}_\eta(0)]$. Since the schedules $K(i)$ and $\mathcal{A}^d(i)$ do not depend on the specification of the bond supply rule $\mathcal{B}(i)$, we have for any such constant $\overline{\mathcal{B}}$ that

$$\overline{\mathcal{B}} + \chi K(\iota) \geq \mathcal{A}^d(\iota) \quad \text{and} \quad \overline{\mathcal{B}} + \frac{\chi(1+\iota)}{1+\chi\iota}K(0) \leq \mathcal{A}^d(0), \tag{B.17}$$

such that, by (36), both an FE and a ZE exist when $\mathcal{B}(i) = \overline{\mathcal{B}}$. ■

B.6 Proof of Proposition 2

Define

$$Z(i) \equiv \mathcal{A}^s(i) - \mathcal{A}^d(i). \tag{B.18}$$

It follows immediately from (36) that a sufficient condition to rule out multiple equilibria is that $Z'(i) < 0$ for $i \in [0, \iota]$. With $\chi = 0$, we have

$$Z(i) = \mathcal{B}(i) - \mathcal{A}^d(i) = \mathcal{B}(i) - \frac{\beta(1+\iota)}{1+i}\theta c_1(i),$$

where we used the definitions of $\mathcal{A}^s(i)$ and $\mathcal{A}^d(i)$ from (33) and (35), respectively. Given $\mathcal{B}'(i) \leq 0$, we thus have that

$$Z'(i) < 0 \quad \Leftrightarrow \quad \frac{\partial \mathcal{A}^d(i)}{\partial i} > 0 \quad \Leftrightarrow \quad \frac{\partial \left[\frac{c_1(i)}{1+i}\right]}{\partial i} > 0 \quad \Leftrightarrow \quad \frac{c_1'(i)}{c_1(i)} > \frac{1}{1+i}. \tag{B.19}$$

With $\chi = 0$, κ does not depend on i (see (28)), which implies that the real wage w does not change with i (see (4)). With CRRA utility, we then have from (27) that

$$c_1 = \left(\frac{1+i}{1+\iota} w\right)^{\frac{1}{\sigma}} \quad \text{and} \quad \frac{c_1'(i)}{c_1(i)} = \frac{1}{\sigma}\frac{1}{1+i}. \tag{B.20}$$

It follows from (B.19) and (B.20) that $Z'(i) < 0$ for $i \in [0, \iota]$ when $\sigma < 1$. ■

B.7 Proof of Proposition 3

Note first that from the definition of $\mathcal{A}^s(i)$ in (33), if $\chi = 0$ and $\mathcal{B}(i) = \overline{\mathcal{B}}$, then $\mathcal{A}^s(i) = \overline{\mathcal{B}}$. Suppose the real bond supply is fixed at $\overline{\mathcal{B}} = \mathcal{A}^d(i_1)$ for some $i_1 \in (0, \iota)$.[27] It then follows from (36) that (i) an IE exists; and (ii) if $\mathcal{A}^d(i)$ is weakly decreasing in i over $i \in [0, \iota]$, then an FE and a ZE exist as well.

Using the definition of $\mathcal{A}^d(i)$ from (35), we have that

$$\frac{\partial \mathcal{A}^d(i)}{\partial i} \leq 0 \quad \Leftrightarrow \quad \frac{\partial \left[\frac{c_1(i)}{1+i}\right]}{\partial i} \leq 0 \quad \Leftrightarrow \quad \frac{c_1'(i)}{c_1(i)} \leq \frac{1}{1+i}. \tag{B.21}$$

From (B.20), we have that with CRRA utility and $\chi = 0$, condition (B.21) is satisfied for all $i \in [0, \iota]$ when $\sigma \geq 1$. Therefore, if $\sigma \geq 1$ and the real bond supply is fixed at $\overline{\mathcal{B}} = \mathcal{A}^d(i_1)$ for some $i_1 \in (0, \iota)$, then an FE, an IE and a ZE coexist.[28] ■

B.8 Proof of Proposition 5

Consider an economy in an IE. From (36), we have that $Z(i) = 0$ in an IE, where $Z(i)$ is defined as in (B.18). It follows from (36) that as ι approaches zero, a sufficient condition for a ZE, an IE and an FE to coexist is that $Z'(i) > 0$ at the point $Z(i) = 0$. In the following, we will derive a sufficient condition for the latter to be the case.

With a constant money-to-debt ratio, the real bond supply satisfies

$$\mathcal{B}(i) = \frac{1-\eta}{\eta}\mathcal{M}(i) = \frac{1-\eta}{\eta}\beta(1+\iota)(1-\theta)c_0(i), \tag{B.22}$$

where, in the second step, we made use of the fact that (30) holds with equality in an IE. Furthermore, with CRRA utility, we have from (19) and (26)-(27) that

$$c_0(i) = \frac{1}{(1-\theta) + \theta(1+i)^{\frac{1}{\sigma}}} C(i) \quad \text{and} \quad c_1(i) = \frac{(1+i)^{\frac{1}{\sigma}}}{(1-\theta) + \theta(1+i)^{\frac{1}{\sigma}}} C(i). \tag{B.23}$$

[27]Since $c_1 > 0$, which follows from the fact that $\lim_{c \to 0} u'(c) = \infty$, $\mathcal{A}^d$ is always strictly positive.

[28]If $\sigma = 1$, there exist a continuum of equilibria, with any $i \in [0, \iota]$ constituting an equilibrium.

Substituting (34), (35), (B.22) and (B.23) into

$$Z(i) = \mathcal{A}^s(i) - \mathcal{A}^d(i) = \mathcal{B}(i) + \frac{\chi(1+\iota)}{1+\chi\iota+(1-\chi)i}K(i) - \mathcal{A}^d(i)$$

yields

$$Z(i) = T(i)C(i), \tag{B.24}$$

where

$$T(i) \equiv \frac{(1-\eta)\beta(1+\iota)(1-\theta)}{\eta[1-\theta+\theta(1+i)^{\frac{1}{\sigma}}]} + \frac{\chi(1+\iota)q(i)}{1+\chi\iota+(1-\chi)i} - \frac{\beta\theta(1+\iota)}{(1-\theta)(1+i)^{1-\frac{1}{\sigma}}+\theta(1+i)}. \tag{B.25}$$

In (B.25), we used the notation $q(i) = q(\kappa(i))$. In an IE, we have $Z(i) = 0$ and, hence, $T(i) = 0$. Evaluating $T(i)$ at the limit where $\iota = i = 0$, we have that

$$T(i) = 0 \quad \Leftrightarrow \quad \eta = \frac{\beta(1-\theta)}{\beta - \chi q(0)} \equiv \hat{\eta}. \tag{B.26}$$

An IE with a strictly positive bond supply exists iff $\hat{\eta} \in (0,1)$; since $q(0) = q(\kappa(0)) = q(\kappa^*) = q^*$ when $\iota = i = 0$, this is equivalent to condition (41) in Proposition 5.

Next, from (B.24), we have that $Z'(i) = T'(i)C(i) + T(i)C'(i)$. In an IE (where $Z(i) = 0$), we therefore have that $Z'(i) > 0 \Leftrightarrow T'(i) > 0$. From (B.25), we find that

$$\begin{aligned} T'(i) = &-\beta(1+\iota)\left(\frac{1-\eta}{\eta}\right)\frac{1}{\sigma}\frac{\theta(1-\theta)(1+i)^{\frac{1}{\sigma}-1}}{\left[1-\theta+\theta(1+i)^{\frac{1}{\sigma}}\right]^2} \\ &+ \frac{\chi(1+\iota)}{1+\chi\iota+(1-\chi)i}q'(i) - \frac{\chi(1-\chi)(1+\iota)}{[1+\chi\iota+(1-\chi)i]^2}q(i) \\ &+ \frac{\beta\theta(1+\iota)}{\left[(1+i)\theta+(1+i)^{1-\frac{1}{\sigma}}(1-\theta)\right]^2}\left[(1-\theta)\left(1-\frac{1}{\sigma}\right)(1+i)^{-\frac{1}{\sigma}}+\theta\right]. \end{aligned} \tag{B.27}$$

Evaluating $T'(i)$ at $\iota = i = 0$ and using the fact that $\eta = \hat{\eta}$ in an IE, we find that

$$T'(i) > 0 \quad \Leftrightarrow \quad \chi(1-\chi)q(0) - \chi q'(0) < \theta\left[\beta - \frac{1}{\sigma}(\beta - \chi q(0))\right]. \tag{B.28}$$

It remains to determine the derivative $q'(0)$. From the definition of $q(i) = q(\kappa(i))$ in (34), we have that

$$q'(i) = \frac{w(\kappa)}{(f(\kappa)-\delta\kappa)^2}\kappa'(i). \tag{B.29}$$

Next, defining

$$F(\kappa, i) = f'(\kappa) + 1 - \delta - \frac{1}{\beta}\frac{1+i}{1+i+\chi(\iota-i)},$$

we have from (28) that $F(\kappa, i) = 0$ implicitly defines κ as a function of i. From the implicit function theorem, we obtain

$$\kappa'(i) = -\frac{F_i'(\kappa,i)}{F_\kappa'(\kappa,i)} = \frac{\chi(1+\iota)}{\beta[1+i+\chi(\iota-i)]^2}\frac{1}{f''(\kappa)}. \tag{B.30}$$

With a CES production function (40) with A=1, we have

$$f'(\kappa) = \alpha \left(\frac{f(\kappa)}{\kappa}\right)^{\frac{1}{\rho}} \quad \text{and} \quad f''(\kappa) = -\frac{w(\kappa)}{\rho}\frac{f'(\kappa)}{\kappa f(\kappa)}. \tag{B.31}$$

Combining (B.29), (B.30) and (B.31) and evaluating $q'(i)$ at $\iota = i = 0$, we obtain

$$-q'(0) = \frac{\chi\rho}{\beta}\frac{\kappa^* f(\kappa^*)}{f'(\kappa^*)(f(\kappa^*) - \delta\kappa^*)^2}. \tag{B.32}$$

Inserting (B.32) into the condition in (B.28) and using $q^* = q(0)$ yields

$$\begin{aligned} & \rho \leq \frac{\beta f'(\kappa^*)(f(\kappa^*) - \delta\kappa^*)^2}{\kappa^* f(\kappa^*)}\left[\frac{\theta}{\chi^2}\left(\beta - \frac{1}{\sigma}(\beta - \chi q^*)\right) - \left(\frac{1}{\chi} - 1\right)q^*\right] \\ \Leftrightarrow \quad & \rho \leq \frac{\beta f'(\kappa^*)}{q^*(1+\delta q^*)}\left[\frac{\theta}{\chi^2}\left(\beta - \frac{1}{\sigma}(\beta - \chi q^*)\right) - \left(\frac{1}{\chi} - 1\right)q^*\right], \end{aligned}$$

which is the same as condition (42) in Proposition 5.

By continuity, conditions (41) and (42) guarantee coexistence of an IE, a ZE and an FE when $\eta = \hat{\eta}$ and $\iota > 0$ is sufficiently close to zero. ■

B.9 Proof of Proposition A.1

The strategy of the proof is constructive: we detail the derivation of the map $X \mapsto X_{-1}$ and then show that this map is unique. We proceed in three steps. In the first step, we show that there is a unique map $(X, \iota, i) \mapsto X_{-1}$, i.e. once we know the nominal rates (ι, i), we can backward iterate on X by one period. In the second step, we show how the nominal rates (ι, i) are related to market clearing in the money market and the asset market. In the third and final step, we show that given X, there is a unique tuple (ι, i) such that $0 \leq i \leq \iota$ (which must always hold to have bounded demand for money and assets) and the money market and the asset market clear.

In the first step, we show that we can determine X_{-1} uniquely from (X, ι, i), were we assume that $0 \leq i \leq \iota$, since this must be true on the equilibrium path. From the resource constraint (2), we have

$$C + K = F(K_{-1}, L) + (1-\delta)K_{-1}. \tag{B.33}$$

Using the CRS property of F, as well as $\kappa \equiv K_{-1}/L$, $C = (1-\theta)c_0 + \theta c_1$, and the first-order conditions for c_0 and c_1 from (23) in the resource constraint, we obtain

$$\begin{aligned} C + K &= L[f(\kappa) + (1-\delta)\kappa] \\ &= K_{-1}[f(\kappa) + (1-\delta)\kappa]/\kappa \\ \Rightarrow \quad K_{-1} &= \frac{C+K}{f(\kappa)/\kappa + 1 - \delta} \\ &= \frac{(1-\theta)c_0 + \theta c_1 + K}{f(\kappa)/\kappa + 1 - \delta} \\ & \frac{(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right) + \theta u'^{-1}\left(\frac{1}{w(\kappa)}\frac{1+\iota}{1+i}\right) + K}{f(\kappa)/\kappa + 1 - \delta}, \end{aligned} \tag{B.34}$$

where $w(\kappa)$ is the wage as a function of κ (see (4)). Expression (B.34) shows that K_{-1} is fully determined by (X, ι, i).

Next, consider the equilibrium net return on capital from (24):

$$\psi(\kappa) + 1 - \delta = \frac{1}{\beta} \frac{w(\kappa)}{w(\kappa_{-1})} \frac{1+i}{(1-\chi)(1+i) + \chi(1+\iota)}, \tag{B.35}$$

where $\psi(\kappa)$ is the rental rate for capital as a function of κ (see (3)). Rearranging terms yields

$$w(\kappa_{-1}) = \frac{1}{\beta} \frac{w(\kappa)}{\psi(\kappa) + 1 - \delta} \frac{1+i}{(1-\chi)(1+i) + \chi(1+\iota)}. \tag{B.36}$$

Since $w(\kappa)$ is strictly increasing in κ, its inverse exists, such that we have

$$\kappa_{-1} = w^{-1} \left(\frac{1}{\beta} \frac{w(\kappa)}{\psi(\kappa) + 1 - \delta} \frac{1+i}{(1-\chi)(1+i) + \chi(1+\iota)} \right), \tag{B.37}$$

which shows that κ_{-1} is fully determined by (X, ι, i).

Finally, recall that $Z \equiv \phi M_{-1}$, such that $Z = Z_{-1}(1+\gamma)/(1+\pi)$. Using the definition of the Fisher rate from (15) implies

$$Z_{-1} = Z \frac{\beta(1+\iota)}{1+\gamma} \frac{w(\kappa_{-1})}{w(\kappa)}. \tag{B.38}$$

Substituting for $w(\kappa_{-1})/w(\kappa)$ using (B.35), we obtain

$$Z_{-1} = \frac{1+\iota}{1+\gamma} \frac{1+i}{(1-\chi)(1+i) + \chi(1+\iota)} \frac{Z}{\psi(\kappa) + 1 - \delta}, \tag{B.39}$$

which shows that Z_{-1} depends only on (X, ι, i), and thus completes the proof that there is a unique map $(X, \iota, i) \mapsto X_{-1}$.

In the second step, we derive conditions on (ι, i) implied by market clearing in the money market and the asset market, for a given X. First, clearance of the market for money requires

$$(1-\theta)c_0 \leq Z, \quad \text{with equality if } i > 0. \tag{B.40}$$

Using the first-order condition for c_0 from (23), this leads to a market clearance condition on (X, ι, i):

$$(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right) \leq Z, \quad \text{with equality if } i > 0. \tag{B.41}$$

Next, clearance of the market for indirectly-liquid assets requires

$$(1-\theta)c_0 + \theta c_1 \leq Z + (1+i)\phi B_{-1} + \chi[\psi(\kappa) + (1-\delta)]K_{-1}. \tag{B.42}$$

Inserting into (B.42) the first-order conditions for c_0 and c_1 from (23), the definition of the Fisher rate from (15), and using our assumption of a fixed bond supply $\overline{\mathcal{B}} = \phi B$ (which implies $\phi B_{-1} = \overline{\mathcal{B}}/(1+\pi)$), we obtain

$$(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right) + \theta u'^{-1}\left(\frac{1}{w(\kappa)} \frac{1+\iota}{1+i}\right)$$

$$\leq Z + \frac{\overline{\mathcal{B}}(1+i)}{\beta(1+\iota)}\frac{w(\kappa)}{w(\kappa_{-1})} + \chi[\psi(\kappa)+1-\delta]K_{-1}, \quad \text{(B.43)}$$

with equality if $i < \iota$. Substituting for K_{-1} and $w(\kappa_{-1})$ by using (B.34) and (B.36), we obtain the asset market clearance condition as a function of (X, ι, i):

$$(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right) + \theta u'^{-1}\left(\frac{1}{w(\kappa)}\frac{1+\iota}{1+i}\right) \leq Z + [\psi(\kappa)+1-\delta] \times \left(\overline{\mathcal{B}}\frac{(1-\chi)(1+i)+\chi(1+\iota)}{1+\iota} + \chi\frac{(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right) + \theta u'^{-1}\left(\frac{1}{w(\kappa)}\frac{1+\iota}{1+i}\right) + K}{f(\kappa)/\kappa + 1 - \delta}\right). \quad \text{(B.44)}$$

Defining

$$\tilde{\psi}(\kappa) \equiv \psi(\kappa) + 1 - \delta \quad \text{and} \quad \tilde{\varphi}(\kappa) \equiv f(\kappa)/\kappa + 1 - \delta, \quad \text{(B.45)}$$

we can write equation (B.44) more compactly as

$$0 \leq Q(X,\iota,i) \equiv Z + \tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{(1-\chi)(1+i)+\chi(1+\iota)}{1+\iota} + \frac{\chi\tilde{\psi}(\kappa)K}{\tilde{\varphi}(\kappa)} - \left[(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right) + \theta u'^{-1}\left(\frac{1}{w(\kappa)}\frac{1+\iota}{1+i}\right)\right]\frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}, \quad \text{(B.46)}$$

with equality if $i < \iota$.[29] Thus, given X, clearance of the money market and the asset market requires that the nominal rates (ι, i) satisfy equations (B.41) and (B.46), and they should satisfy $0 \leq i \leq \iota$.

The third and last step is to show that given X, there is a unique (ι, i) such that $0 \leq i \leq \iota$ and equations (B.41) and (B.46) are satisfied. We split this part up by considering two equilibrium cases separately: (a) $0 < i \leq \iota$ and (b) $i = 0$.

Case (a): $0 < i \leq \iota$. With $i > 0$, condition (B.41) implies

$$(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right) = Z \quad \Rightarrow \quad 1+\iota = w(\kappa)u'\left(\frac{Z}{1-\theta}\right). \quad \text{(B.47)}$$

Since $i > 0$ implies $\iota > 0$, we thus obtain an equilibrium condition

$$Z < (1-\theta)u'^{-1}\left(\frac{1}{w(\kappa)}\right), \quad \text{(B.48)}$$

which depends only on X.

Note next that equation (B.47) pins down ι as a function of X. We can thus reformulate $Q(X, \iota, i)$ from (B.46) as

$$Q(X,\iota(X),i) = \frac{\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}Z + \tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{(1-\chi)(1+i)+\chi(1+\iota(X))}{1+\iota(X)} + \frac{\chi\tilde{\psi}(\kappa)K}{\tilde{\varphi}(\kappa)}$$

[29]Note that we have $\tilde{\psi}(\kappa) < \tilde{\varphi}(\kappa)$ since $f(\kappa)$ is strictly concave.

$$-\theta\frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}u'^{-1}\left(\frac{1}{w(\kappa)}\frac{1+\iota(X)}{1+i}\right). \quad \text{(B.49)}$$

Taking the partial derivative of $Q(X,\iota(X),i)$ w.r.t. $1+i$, we find

$$\frac{\partial Q(X,\iota(X),i)}{\partial(1+i)}=\tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{1-\chi}{1+\iota(X)}+\theta\frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}\frac{u'(c_1)}{u''(c_1)}\frac{1}{1+i}. \quad \text{(B.50)}$$

Using that $cu''(c)/u'(c)>-1$, we thus have

$$\begin{aligned}\frac{\partial Q(X,\iota(X),i)}{\partial(1+i)}&<\tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{1-\chi}{1+\iota(X)}-\theta\frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}\frac{u'^{-1}\left(\frac{1}{w(\kappa)}\frac{1+\iota(X)}{1+i}\right)}{1+i}\\&=-\frac{1}{1+i}\left[\frac{\chi\tilde{\psi}(\kappa)[Z+K]}{\tilde{\varphi}(\kappa)}+\chi\tilde{\psi}(\kappa)\overline{\mathcal{B}}-Q(X,\iota(X),i)\right].\end{aligned} \quad \text{(B.51)}$$

Since $i\leq\iota$ and $Q(X,\iota(X),i)\geq 0$, with equality if $i<\iota$, it follows directly that an $i\leq\iota(X)$ which solves $Q(X,\iota(X),i)\geq 0$ (with equality if $i<\iota(X)$) is unique. Further, it exists and satisfies $i>0$ iff $Q(X,\iota(X),0)>0$, which in turn holds iff

$$\frac{[\chi\tilde{\psi}(\kappa)-\theta\tilde{\varphi}(\kappa)]Z+\chi\tilde{\psi}(\kappa)(1-\theta)K}{(1-\theta)\tilde{\varphi}(\kappa)}+\tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{1-\chi+\chi(1+\iota(X))}{1+\iota(X)}>0. \quad \text{(B.52)}$$

Using (B.47) to substitute for $\iota(X)$, we obtain

$$\frac{[\chi\tilde{\psi}(\kappa)-\theta\tilde{\varphi}(\kappa)]Z+\chi\tilde{\psi}(\kappa)(1-\theta)K}{(1-\theta)\tilde{\varphi}(\kappa)}+\tilde{\psi}(\kappa)\overline{\mathcal{B}}\left[\frac{1-\chi}{w(\kappa)u'\left(\frac{Z}{1-\theta}\right)}+\chi\right]>0, \quad \text{(B.53)}$$

which yields an equilibrium condition that depends only on X.

Thus, given X, a tuple (ι,i) with $0<i\leq\iota$ that solves equations (B.41) and (B.46) exists iff

$$X\in\mathscr{X}_{i>0}\equiv\left\{X\in\mathbb{R}^3_+\quad\text{s.t. (B.48) and (B.53)}\right\}, \quad \text{(B.54)}$$

and this (ι,i) is pinned down uniquely as a function of X.

Case (b): $i=0$. With $i=0$, we obtain from (B.46) that

$$\begin{aligned}0\leq Q(X,\iota,0)=Z+\tilde{\psi}(\kappa)\overline{\mathcal{B}}&\frac{1-\chi+\chi(1+\iota)}{1+\iota}\\&+\frac{\chi\tilde{\psi}(\kappa)K}{\tilde{\varphi}(\kappa)}-u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right)\frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)},\end{aligned} \quad \text{(B.55)}$$

with equality if $\iota>0$. Taking the partial derivative of $Q(X,\iota,0)$ w.r.t. $1+\iota$ yields

$$\frac{\partial Q(X,\iota,0)}{\partial(1+\iota)}=-\tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{1-\chi}{(1+\iota)^2}-\frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}\frac{u'(C)}{u''(C)}\frac{1}{1+\iota}. \quad \text{(B.56)}$$

Using that $cu''(c)/u'(c)>-1$, we thus have

$$\begin{aligned}\frac{\partial Q(X,\iota,0)}{\partial(1+\iota)}&>-\tilde{\psi}(\kappa)\overline{\mathcal{B}}\frac{1-\chi}{(1+\iota)^2}+\frac{\tilde{\varphi}(\kappa)-\chi\tilde{\psi}(\kappa)}{\tilde{\varphi}(\kappa)}u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right)\frac{1}{1+\iota}\\&=\frac{Z+\chi\tilde{\psi}(\kappa)K/\tilde{\varphi}(\kappa)+\tilde{\psi}\chi\overline{\mathcal{B}}-Q(X,\iota,0)}{1+\iota}.\end{aligned} \quad \text{(B.57)}$$

Since $\iota \geq 0$ and $Q(X, \iota, 0) \geq 0$ (with equality if $\iota > 0$), it follows directly that an $\iota > 0$ which solves $Q(X, \iota, 0) \geq 0$ (with equality if $\iota > 0$) is unique. Equilibrium existence, however, requires that equation (B.41) holds, i.e.

$$(1-\theta)u'^{-1}\left(\frac{1+\iota}{w(\kappa)}\right) \leq Z \quad \Rightarrow \quad 1+\iota \geq w(\kappa)u'\left(\frac{Z}{1-\theta}\right). \tag{B.58}$$

Since the $\iota \geq 0$ which solves $Q(X, \iota, 0) \geq 0$ (with equality if $\iota > 0$) is non-negative, condition (B.58) is satisfied trivially if

$$Z \geq (1-\theta)u'^{-1}\left(\frac{1}{w(\kappa)}\right), \tag{B.59}$$

which is the exact opposite of condition (B.48). If condition (B.59) does not hold, then condition (B.58) is satisfied if and only if $Q(X, \underline{\iota}, 0) > 0$, where $1+\underline{\iota} \equiv w(\kappa)u'\left(\frac{Z}{1-\theta}\right)$. This translates into

$$\frac{[\chi\tilde{\psi}(\kappa) - \theta\tilde{\varphi}(\kappa)]Z + \chi\tilde{\psi}(\kappa)(1-\theta)K}{(1-\theta)\tilde{\varphi}(\kappa)} + \tilde{\psi}(\kappa)\overline{\mathcal{B}}\left[\frac{1-\chi}{w(\kappa)u'\left(\frac{Z}{1-\theta}\right)} + \chi\right] \leq 0, \tag{B.60}$$

which is the exact opposite of Condition (B.53).

Thus, given X, a tuple (ι, i) with $0 = i \leq \iota$ that solves equations (B.41) and (B.46) exists iff

$$X \in \mathscr{X}_{i=0} \equiv \left\{X \in \mathbb{R}^3_+ \quad \text{s.t. (B.59) or (B.60)}\right\}, \tag{B.61}$$

and this (ι, i) is pinned down uniquely as a function of X.

From our discussion of the cases (a) and (b), we see that $\mathscr{X}_{i=0}$ is the complement of $\mathscr{X}_{i>0}$. Hence, given X, we find a unique (ι, i) such that: (i) $0 \leq i \leq \iota$; (ii) the money market clears, i.e. condition (B.41) holds; (iii) the asset market clears, i.e. condition (B.46) holds. In other words, there is a unique map $X \mapsto (\iota, i)$. Since we have also established that there is a unique map $(X, \iota, i) \mapsto X_{-1}$, this means that there is a unique map $X \mapsto X_{-1}$. ■

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