Issues Concerning Determinacy, Learnability, Plausibility, and the Role of Money in New Keynesian Models

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ABSTRACT

In recent monetary policy analysis, it has been common practice to view models as possessing *determinacy* if they feature a single rational expectations (RE) solution that is dynamically stable. Cochrane (2007) has emphasized that the single-stable solution (SSS) condition is not sufficient as a criterion of determinacy, however, because in typical New Keynesian models when the Taylor Principle is satisfied there exists a dynamically explosive solution that is not ruled out by any transversality condition and accordingly can be eliminated only by an arbitrary dictum. McCallum (2009b) has agreed with this specific proposition, but has shown that in these models it is typically the case that the explosive solutions in question are not least-squares learnable [a la Evans and Honkapohja (2001)]. Further, he has argued that such learnability should be considered a *necessary* condition for a solution to be regarded as a model's prediction of the depicted economy's behavior since it amounts to a *feasibility* condition that pertains to information available to individual agents. Cochrane (2009) contends that there are three weaknesses in McCallum's argument, but McCallum (2009c) claims that in all three cases Cochrane's argument is analytically incorrect. The present paper reviews and exposits this exchange and briefly reviews two additional criteria for determinacy, one of McCallum's that rules out solutions with infinite discontinuities and one by Cho and Moreno (2011) that promotes a unique "bubble-free" solution obtained by convergent iterative substitution into the future.

1. Introduction

It is very well known that linear rational expectations models typically possess more than one solution.[1] Instead, there is typically a multiplicity of relationships from which time paths for endogenous variables are predicted as the response of the model economy to given behavioral relations and given processes that generate exogenous variables. This is an unsatisfactory state of affairs since it implies that the model at hand does not provide a unique prediction about how the model economy—and thus the actual economy being modeled—behaves. Consequently, a substantial number of additional requirements have been proposed by various researchers in order to obtain a unique prediction of the analysis to the question of how the economy would behave under specified policies—a prediction that is the *raison d'etre* of a monetary (or non-monetary!) policy model.

In monetary economics in particular one criterion has been rather widely accepted. It is the criterion of a single stable solution (SSS), i.e., that among the multiple solutions that satisfy all of the model's relationships, plus the orthogonality implications of rational expectations (RE), there is one and only one that is dynamically stable (non-explosive). This criterion is in practice often referred to as "determinacy," as if the SSS requirement was equivalent to the desired condition—namely, that the model at hand provides a unique prediction as to the behavior of the (model) economy. A unique prediction is what "determinacy" is supposed to *mean*, however, so it is unsatisfactory for this word to be used as a synonym for the SSS condition. This point has been made implicitly but effectively by Cochrane (2007), who has argued that in a wide class of "New Keynesian" (NK) models (which have been the centerpiece of monetary analysis over the past two decades) policy behavior satisfying the "Taylor Principle" leads to satisfaction of the

[1] Of course the situation is even more unfavorable in non-linear models. In monetary policy analysis the usual procedure is to work with linear models that may be regarded as approximations to actual structures.

SSS requirement but nevertheless features the existence of an explosive path that is not ruled out by any transversality condition or any other generally accepted economic principle. Thus the SSS solution does not provide a unique prediction as to the behavior of the model economy, leading Cochrane to argue that NK analysis is fundamentally flawed.[2] McCallum (2009b) agrees with Cochrane's analytical point regarding the explosive path but shows that the SSS solution does, and the explosive solution does not, satisfy the criterion of "least squares learnability." Furthermore, McCallum argues, this type of learnability should be considered as a necessary condition for a solution to be viewed as a plausible contender for the unique prediction of behavior provided by the model at hand. Cochrane (2009) contends, however, that the analysis in McCallum (2009b) is flawed in three ways. In response to this, McCallum (2009c) explains that Cochrane's objections are analytically incorrect or inapplicable. In the present paper I will review this exchange concerning fundamental issues in monetary policy analysis. In addition, Sections 5 and 6 of the paper will consider the possible usefulness of two very recently developed candidates—one by Cho and Moreno (2011) and one by myself (2011b)—as the "selection criterion" for the designation of a particular solution as the one that is relevant for practical analysis with a linear RE model.

2. Cochrane's Challenge

A standard three-equation NK model might be written as

(1) $y_t = b_0 + b_1(R_t - E_t\pi_{t+1}) + E_t y_{t+1} + v_t \qquad b_1 < 0$

(2) $\pi_t = \beta E_t\pi_{t+1} + \kappa(y_t - \bar{y}_t) + u_t \qquad \kappa > 0$

(3) $R_t = \mu_0 + (1 + \mu_1)\pi_t + \mu_2(y_t - \bar{y}_t) + e_t$

[2] Somewhat related arguments have been expressed by Minford and Srinivasan (2009)(2011).

where the variables are y_t = output gap, π_t = inflation rate, and R_t = one-period interest rate, all expressed in terms of fractional deviations from steady-state values. Here (1) is an "expectational IS equation" that combines the intertemporal Euler equation (for a typical infinite-lived household with standard time-separable preferences) with a linearized overall resource constraint plus the assumption that the economy-wide capital stock is fixed. Equation (2) is the standard Calvo model of imperfectly flexible price setting, and (3) is the central bank's policy rule that specifies settings of the interest instrument R_t in response to current values of inflation and the output gap. With $\mu_1 > 0$ the Taylor Principle will be satisfied even if $\mu_2 = 0$. The exogenous shocks in this system v_t , u_t, and e_t are, respectively, shocks pertaining to (i) time-preference plus natural-rate of output plus government consumption fluctuations, (ii) price-setting behaviour, and (iii) monetary policy behavior. This is a simplified setup but is highly representative of current mainstream analysis; furthermore there will be no loss in substance relating to the issues at hand if we simplify even farther by supposing (i) that prices are fully flexible, in which case (2) becomes $y_t = \bar{y}$, and also (ii) setting setting $\bar{y} = 0$. Then we can write the system as

(1') $$0 = b_0 + b_1(R_t - E_t\pi_{t+1}) + v_t$$

(3') $$R_t = \mu_0 + (1 + \mu_1)\pi_t + e_t$$

and can combine these two relations to yield

(4) $$0 = b_0 + b_1[\mu_0 + (1+\mu_1)\pi_t + e_t - E_t\pi_{t+1}] + v_t$$

The latter is the form of the model that is discussed by Cochrane (2007, pp. 5-10), McCallum (2009b), and Cochrane (2009).

Quantitatively, most analysts would expect the variability of v_t to exceed that of e_t, but I will follow Cochrane (2007) in neglecting the former, a step that is substantively unjustified but innocuous in terms of the issues at hand. For the policy shock, it is assumed that an AR(1) process obtains, i.e., that $e_t = \rho e_{t-1} + \varepsilon_t$ with ε_t being white noise and with $|\rho| < 1$. With this setup, it is natural to conjecture that there will be a solution of the form

(5) $\pi_t = \phi_0 + \phi_1 e_t$

with expectations therefore obeying $E_t\pi_{t+1} = \phi_0 + \phi_1\rho e_t$. Substitution in (4) then implies that this solution is

(6) $$\pi_t = 0 - \frac{1}{1+\mu_1-\rho} e_t .$$

The latter is often referred to as the "fundamentals" or "MSV" solution, its identifying characteristic being that it does not include state variables other than those required by the structural model. With $\mu_1 > 0$ and $|\rho| < 1$, (6) implies that π_t is negatively related to e_t and that larger values of μ_1 serve to reduce the variability of π_t around its target.

Now suppose, however, that instead of (5) one looks for a solution of the form

(7) $\pi_t = \phi_0 + \phi_1 e_t + \phi_2\pi_{t-1}$.

Then $E_t\pi_{t+1} = \phi_0 + \phi_1\rho e_t + \phi_2(\phi_1 e_t + \phi_2\pi_{t-1})$ and a second solution, in addition to (6), is

(8) $\pi_t = (1/\rho)e_t + (1+\mu_1)\pi_{t-1}$.

Clearly, with $\mu_1 > 0$, as specified by the Taylor Principle, this expression (8) implies an explosive process for the inflation rate.[3] That is the type of solution noted by Cochrane and referred to above. It seems clear that in the model at hand there is no transversality condition that would rule out this explosive solution for the inflation rate, and that this is the case is verified explicitly

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[3] It appears from (8) that this solution will not be defined in the measure-zero case with $\rho = 0$. But in that case one can add e_{t-1} as an additional state variable in (7) and obtain an infinity of explosive solutions that could be indexed by the start-up value of π_{t-1}.

on pp. 1106-1107 of McCallum (2009b). Accordingly, one must agree that Cochrane is right to argue that, although there is a single stable solution in the case at hand, there is not—as matters stand—"determinacy" *in the sense of a unique solution* consistent with the model.

My position, however, is that we should not be satisfied with "as matters stand," for the solution permitted by (8), rather than (6), is not least-squares learnable in the manner emphasized by Evans and Honkapohja (1999) (2001)—this is demonstrated in McCallum (2009b). As a consequence, I suggest that the alternative solution (8) is not plausible—and thus that (6) is in fact the only plausible outcome predicted by the model.

3. Learnability and Feasibility

What is the nature of the postulated implausibility implied by the absence of least-squares learnability in the case of the alternative solution (8)? It should arguably be termed an *infeasibility*, albeit one that pertains to information rather than (for example) a standard production function. To see in what sense this strong claim is viable one must consider the nature of least-squares learnability (henceforth LSL). LSL or its absence is a property, in the context of a specified model, of each particular RE solution.

Here I will proceed by arguing in effect that LS learnability [as defined by, e.g., E&H (2001, pp. 200, 232-233)] should be regarded as a part of the requirement for a RE equilibrium. The motivation for this additional requirement is simple. First, for expectations to meet the orthogonality requirements for RE, the agents must have considerable quantitative, as well as qualitative, information concerning the economy's behavior. Second, in any RE model intended to represent behavior in an actual market economy, the individual agents should be able to learn these quantitative details concerning the behavior of variables—which they must forecast for decision-making purposes—from data generated by the economy itself.[4] Certainly they cannot obtain such data by introspection, magic, or divine revelation.

The first task at hand is to describe, in a reasonably precise way, the learning process that is being discussed. Rather than beginning with the highly special model summarized by equation (4), I will proceed in terms of a rather general linear framework that will give a better idea of the nature and scope of the learning process without requiring any significant complication in the presentation.[5] In this discussion, I assume that we are concerned with an economy in which the agents are alike, but behave entirely independently. Suppose that the behavior of per-capita values of prices and quantities is given by

(9) $$x_t = AE_t x_{t+1} + Cx_{t-1} + Dz_t$$

where x_t is a m×1 vector of endogenous variables, while exogenous variables z_t (n×1) are

(10) $$z_t = Rz_{t-1} + \varepsilon_t$$

with ε_t white noise and R being a stable matrix. Considering fundamental solutions of form

(11) $$x_t = \Omega x_{t-1} + \Gamma z_t,$$

standard undetermined-coefficient reasoning establishes that, with RE temporarily specified, the matrices Ω and Γ must satisfy

(12a) $$A\Omega^2 - \Omega + C = 0$$

(12b) $$\Gamma = A\Omega\Gamma + A\Gamma R + D.$$

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[4] For RE to obtain, the implied forecasting relationships must be quantitatively accurate. The statement in this sentence is part of a rationale for my position concerning learnability; it is not intended to serve as a definition of learnability. In addition it should be noted that the present paper is concerned throughout with RE equilibria that are conventional except with respect to the informational aspect just mentioned. In particular, the paper does not address game-theoretic issues such as those implied by monetary policy actions that depart systematically from a specified policy rule.

[5] A short presentation in terms of the special model (4) is included as appendix A.

For a given Ω, the matrix Γ will be determined uniquely—but there are many matrices Ω that satisfy (12a). Furthermore, if more than one of them has all its eigenvalues smaller than 1.0 in modulus there are multiple stable solutions.

Now, for RE to prevail, agents need to base their expectations on accurate quantitative knowledge of Ω and Γ; so what the agents need to learn about is the system's law of motion. Such knowledge cannot, in reality and therefore in a satisfactory model, be obtained by introspection or divine revelation, but must be determined from data generated in the past by the economy itself. The LS learning process for acquiring such knowledge is as follows. In period t, agents obtain estimates Ω_t and Γ_t (of Ω and Γ) by ordinary least squares using past data: they estimate the relationship $x_\tau = \Omega_t x_{\tau-1} + \Gamma_t z_\tau$ using data from periods dated τ = t−1 and earlier. Using these estimates, the agents forecast x_{t+1} as $x^e_{t+1} = \Omega_t x_t + \Gamma_t R z_t$.[6] [7] Then from substitution in (9), but using x^e_{t+1} in place of $E_t x_{t+1}$, the outcome x_t is generated—as a consequence of the supply and demand behavior summarized in (9)—as

(13) $$x_t = A[\Omega_t x_t + \Gamma_t R z_t] + C x_{t-1} + D z_t .$$

Next, in period t+1 agents add the newly generated observation to their data, estimate Ω_{t+1} and Γ_{t+1}, form expectations x^e_{t+2}, make supply-demand choices via (9), and observe x_{t+1}. Then in periods t+2, t+3, ... the process continues.[8]

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[6] The discussion proceeds for simplicity as if R were known. E&H (2001, p. 181) mention, however, that this assumption is not needed for the relevant results (since exogenous variables can be treated as endogenous). It might be mentioned incidentally that in the discussion of E&H (2001) the least squares estimation calculations are often described as being conducted by recursive least squares.

[7] It is here being assumed that x_t and z_t are observed by agents in period t. It is possible, however, that the model specifies that expectations (and other determinants of actions in t) relevant in t are based on data only from earlier time periods.

[8] Evans and Honkapohja (2001) also consider a second information assumption, namely, that in period t an observation on x_t is not available in the learning process. In this case the term x_t on the right-hand side of (13)

We visualize this process as going on indefinitely. Then, for a particular value of Ω (with associated Γ) we can ask if the process is "locally stable" in the sense that the estimates Ω_t and Γ_t approach the (hypothetically) true values Ω and Γ as $t \to \infty$, provided that they begin in the "first period" with estimates that are not too far from these true values. If the answer is "yes," then that particular RE solution is "stable under LS learning," in the Evans and Honkapohja terminology, and the model economy can be viewed as tending to behave as implied by that RE solution if it (the economy) has been operating for a large number of periods. If, on the other hand, a particular RE solution, relating to a particular solution to (12a) for Ω, is not stable under LS learning, the implication/prediction of the analysis is that the model economy—and thus the actual economy represented by the model—will not be found in an equilibrium corresponding to this particular RE solution.[9]

Under what conditions will a particular RE solution, itself dynamically stable, be stable under LS learning? For the model (9)-(10), the basic analytical result of Evans and Honkapohja is, in my notation, that the LS learning process will be locally stable if all of the eigenvalues of the following matrices have real parts smaller than 1.0:

(14a) $F = (I - A\Omega)^{-1} A$

(14b) $\Omega' \otimes F$

(14c) $R' \otimes F$.

The proof of this result is presented by E&H (1998) on pp. 26-32, the last two pages being an application to the linear framework of equations (9)-(11) of a more general analysis [by

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would be replaced with x_t^e. In the monetary model under discussion in Section 2, this would result in no change in the learnability findings—although it will in cases in which $C \neq O$ in (9). See E&H (2001, pp. 244-245).

[9] It is the case that in some of their work E&H use a "constant gain" version of the LS learning process as a model of expectation formulation, rather than as a justification of a particular RE solution. See e.g., E&H (2001, pp. 333-359). Analysts following this approach include Orphanides and Williams (2005), among others.

Benveniste, Métivier, and Priouret (1990)] that is summarized on E&H's pp. 26-30. Alternatively, the result is presented in the treatise of E&H (2001), with the summary statement of pp. 236-238 utilizing analysis developed on pp. 229-235 and 121-134. If, on the other hand, conditions (14) are not satisfied, E&H indicate that, with a few regularity assumptions, "… one can show convergence [obtains] with probability zero" (1998, p. 32).

At this point a critic might object on the grounds that there are many possible learning processes, and one cannot know that the one described above is realistic. My response to that observation emphasizes that LS learnability is being used, in the present argument concerning monetary policy, as a <u>necessary</u> condition for plausibility of a RE equilibrium. In that context I have argued before that: "... Of course any particular learning scheme might be incorrect in its depiction of actual learning behavior. But in this regard it is important to note that the LS learning process in question assumes that (i) agents are collecting an ever-increasing number of observations on all relevant variables while (ii) the structure is remaining unchanged. Furthermore, (iii) the agents are estimating the relevant unknown parameters (iv) with an appropriate estimator in (v) a properly specified model. Thus if a proposed RE solution is not learnable by the process in question—the one to which the E&H results pertain—then it would seem highly implausible that it could prevail in practice" (McCallum, 2007, p. 1378). In other words, the process is distinctly "slanted" or "biased" toward a finding of learnability, a bias that is suitable for a necessary condition.

It might be countered that the notion being advanced—that individual agents should have some way of obtaining the information necessary to form expectations—represents an additional requirement not included in standard definitions of economic equilibria. That is so, but in standard (i.e., RE) analysis, for the determination of each period's endogenous variables, one specifies information sets that include quantitative features of the system plus, at a maximum, current and past values of relevant variables.[10] In stochastic models, therefore, knowledge of future variables is excluded as infeasible. In that manner, standard analysis does typically include a form of information feasibility as a requirement for equilibrium. What LS learnability does, in a sense, is to extend the requirement of information feasibility so as to pertain to some limited quantitative knowledge of the economy's structure. More specifically, for agents to be able to form expectations rationally—i.e., without systematic expectational errors—they must be able to develop quantitative knowledge of the economy's law of motion on the basis of observations from its past. Then LS learnability posits a particular process of information acquisition that is highly optimistic with respect to the possibility of the requisite information being acquired. It is in this sense, that the absence of LS learnability may be regarded as representing a type of informational infeasibility.

With regard to monetary policy, the point of the present discussion is, of course, that application of the LS learnability criterion does, in the monetary policy model of Section 2, support the standard NK solution (6) while eliminating the explosive solution of equation (8). That conclusion was established by the general analyses of Bullard and Mitra (2002) and Honkapohja and Mitra (2004), and was mentioned by McCallum (2003, p. 1161), but let us verify it here by reference to the setup of equations (1)-(6) and conditions (14).

To begin, it will be noted that, if there are no lagged endogenous variables in the system, then C = 0 implying that Ω = 0 and F = A. Then the first two of conditions (14) amount to the requirement that the eigenvalues of A all have real parts less than 1. In the basic system

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[10] For some purposes perfect-foresight analysis is useful, of course, but one would not use that assumption in an analysis that is concerned with (e.g.) the variability of asset prices or macroeconomic variables in a setting that includes stochastic shocks.

summarized in (5) above, the fundamentals solution has $\Omega = 0$ and $A = 1/(1 + \mu_1)$. Thus it is clear that for solution (6) the LS learnability requirements (14a-c) are satisfied. By contrast, the non-fundamentals solution (8) yields $\pi_t = (1/\rho)e_t + (1+\mu_1)\pi_{t-1}$, implying that $\Omega = 1 + \mu_1$ and thus that $F = (I - A\Omega)^{-1}A = [1-((1+\mu_1)/(1+\mu_1))]^{-1}(1/(1+\mu_1)) = [1\text{-}1]^{-1}(1/(1+\mu_1)) = (1/(1+\mu_1))/0$. Thus in this case $F > 1 + \mu_1$ (an understatement) and at least two of the three conditions (14) are violated. Accordingly, the explosive solution is not learnable—convergence to (8) occurs with probability zero. Although (8) satisfies the orthogonality conditions for a RE solution, it is implausible (according to the present argument) that an economic system matching the model's specification would generate outcomes of the type that (8) describes.

4. Cochrane's Objections

As already mentioned, Cochrane does not accept the conclusion presented in the previous section. He does not explicitly disagree with the idea that learnability is important, but presents three objections to my application in the monetary policy context that is at issue. These objections are presented in his JME comment (Cochrane 2009); they will be reviewed and refuted in the present section.

First, and most prominently, Cochrane argues that the monetary policy shock—e_t in equations (3), (6), and (8) above—is not observable by private agents but is implicitly assumed to be so in my analysis. His unobservability concern is reasonable, I would agree, and it is true that I did not take any explicit account of that point in my argument in McCallum (2009b). It transpires, however, that this does not matter for the issue at hand—i.e., learnability vs. non-learnability of the NK and explosive RE solutions featured by Cochrane. A preliminary observation is that in the monetary policy case of concern, he and I should probably both be including the private-sector technology shock as the relevant one in the stripped-down, one-shock model in which the current discussion is being conducted.[11] But, putting aside that matter, the main point is that the absence of observability of e_t does not invalidate my paper's argument regarding learnability vs. non-learnability in the analysis at hand. Let us again consider the model in (5′) of McCallum (2009b), namely, $\pi_t = aE_t\pi_{t+1} + u_t$ with $u_t = -ae_t$ and $e_t = \rho e_{t-1} + \varepsilon_t$ (ε_t is white noise and a<1). Thus the model can be written as

$$\pi_t = \rho\pi_{t-1} + aE_t\pi_{t+1} - a\rho E_{t-1}\pi_t - a\varepsilon_t$$

where the only unobservable component is white noise. Then we can define $\xi_t = E_t\pi_{t+1}$ and formulate the model as

$$(15) \qquad \begin{bmatrix}\pi_t\\ \xi_t\end{bmatrix} = \begin{bmatrix}a & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}E_t\pi_{t+1}\\ E_t\xi_{t+1}\end{bmatrix} + \begin{bmatrix}\rho & -a\rho\\ 0 & 0\end{bmatrix}\begin{bmatrix}\pi_{t-1}\\ \xi_{t-1}\end{bmatrix} + \begin{bmatrix}-a\varepsilon_t\\ 0\end{bmatrix}$$

If we use A and C to denote the two 2×2 matrices, does this system have the same learnability conditions as given for the system (4) and $e_t = \rho e_{t-1} + \varepsilon_t$ above? I have not been able to find any results in Evans and Honkapohja (E&H) (2001) that apply to this particular formulation,[12] but in E&H (1998, pp. 30-32) there is an applicable analysis, and it indicates that the presence of an unobserved white noise shock is irrelevant to learnability of the various RE solutions.[13] Specifically, the relevant mapping (from perceived to actual law of motion) does not involve parameters relating to the unobservable shock. This result evidently pertains to all models in the

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[11] It is a more appropriate shock to include because we are concerned with economies in which technology shocks certainly occur, whereas exogenous policy shocks should not appear at all in a well-designed monetary policy rule. Then the shock in the system being analyzed would be the technology shock for the typical agent (v_t in my paper's equation (1)) and would be observable.

[12] There are numerous examples in E&H (2001) in which unobservable white noise shocks are included and do not overturn results that obtain in their absence, but unobservable shocks that are autocorrelated are not considered. In the analysis of systems such as (9)(10) of my paper, there are no white-noise or other unobservables.

[13] In addition, George Evans has provided me with an explicit proof that, in the case at hand, the fundamentals solution is learnable and the alternative explosive solution is not learnable. His note is available on the web at: http://public.tepper.cmu.edu/facultydirectory/FacultyDirectoryProfile.aspx?id=96

very broad class implied by the formulation in equations (9) and (10) above. One way to understand this perhaps surprising result is as follows: Exogenous white noise shocks do not overturn learnability because they represent purely random influences that are overwhelmed asymptotically as the number of observations (on an unchanging system) increases. Furthermore, with shocks that are not white noise, the systematic component can be taken care of by way of its effects on observable variables, as in equation (1). This is not strictly possible for moving-average shocks, rather than autoregressive shocks, but one can approximate moving-average shocks with autoregressive shocks of a larger-than-usual but finite order.[14]

The second topic in Cochrane's criticism concerns his interpretation of the monetary policy rule, which he describes as involving central-bank "hyperinflationary threats" or threats to "blow up the world" (Cochrane 2007, p. 4). In this regard his position seems untenable. In the model that we are discussing, monetary policy is specified by the rule $R_t = \mu_0 + (1 + \mu_1)\pi_t + e_t$ in all circumstances. If "threats" were relevant there would be threatened departures from this rule, to be invoked in certain specified situations. Now, there may be good reasons to be interested in game-theoretic analyses involving alternative modes of central-bank behavior. But in the basic rational-expectations analysis of the model that is under discussion there is nothing of that type. What the central bank's rule promises is to make nominal interest rates higher than otherwise when inflation is above its target value. In the presence of some price-level stickiness—not included in our stripped-down models but included in the three-equation NK framework that both Cochrane and I would use if there were any essential difference in outcome—the higher nominal interest rate would then bring about a reduced level of real aggregate demand. Thus the policy behavior promised by the Taylor rule is to make demand relatively low when inflation

[14] This issue arises also with observable shocks.

(and/or inflationary pressure) is relatively high.

As a final matter, there are numerous references in Cochrane's discussion to the lack of identification of the policy parameter 'a' (equivalent to $1/(1+\mu_1)$ above). Here Cochrane's analysis fails to recognize the crucial difference between structural and reduced-form relationships in the context of learnability analysis. The point is that that learnability of the type under discussion does not require identification, by the model economy's agents, of this parameter. The way that the relevant type of learnability analysis proceeds is by substituting forecasts from a vector-autoregression model (estimated on all past observations) in place of expectational variables in the model, and then determining whether the implied behavior converges to a particular RE solution as time passes and data used in estimation of the forecasting regression increases. The relations estimated for learning purposes by agents in the model are therefore reduced-form, not structural, equations (from agents' perspective). So the non-identification of the structural parameter in question is not relevant to the issue of learnability.

At this point I turn to more recent developments concerning these arguments. First, in a revised version of his NBER working paper 13409, issued (with the same number) in 2010, Cochrane argues explicitly that identifiability of the parameter in question (e.g., a or μ_1) would be important in a *researcher's* empirical study designed to determine whether the policy rule followed by an actual central bank (over some span of time) did, or did not, satisfy the Taylor-Principle requirement (which is that $\mu_1 > 0$ in the simple system being discussed above). That point is, evidently, correct. Accordingly, I must express basic agreement with Cochrane's belief that studies such as Clarida, Gali, and Gertler (2000) have not correctly located the nature of the actual weakness in U.S. monetary policy during the 1970s. I am not entirely convinced, however, that the necessary identification would be absent in more fully developed models, in which there might be relevant predetermined variables in the system that are plausibly excluded from the monetary policy rule. More generally, there are various points developed in this revised version of WP 13409, but it does not consider the role of learnability.

More recently, a slightly revised version of this working paper, entitled "Determinacy and Identification with Taylor Rules," has appeared in the Journal of Political Economy (Cochrane, 2011). My first impression is that I am in substantial agreement with most of the arguments—to the extent that I understand them—in this published paper. It, however, includes no consideration at all of learnability. But an appendix to this paper, Appendix B, is available on the JPE web site and on Cochrane's home page. This appendix does discuss learnability and other aspects of my analysis. Unfortunately, it repeats the mistakes mentioned above: (i) a mistaken belief that the non-observability by agents of the monetary policy shock (e_t in equations (1)-(8) above) overturns the learnability finding when the Taylor principle is satisfied and (ii) the mistaken belief that identification of the policy-rule parameter 'a' by agents in the model is needed for learnability.[15] Thus, my argument above continues to apply.

5. Alternative Selection Criteria: Cho and Moreno

The preceding sections have suggested that LS learnability should be considered as a necessary condition for a specific RE solution to be considered economically plausible. In this section and the next the object is to relate that suggestion to two recent proposals for a "selection criterion" to be used in designating which of a model's multiple RE solutions should be regarded as providing its predictions concerning economic behavior. In an imaginative and promising recent publication, Cho and Moreno (2011) have developed an algorithm for generating a

[15] In addition, incidentally, Cochrane incorrectly characterizes my (2003) distinction between "nominal indeterminacy" and the real solution multiplicity with whch he and I are both concerned in the matters at hand.

"forward solution" of the fundamentals type (i.e., excluding "bubble" components) that, provided that it exists, is unique for all cases of the general linear model specification given above in equations (9) (10). A brief outline is as follows. Given (9) and (10), one can define the following matrices: $M_1 = A$, $\Omega_1 = C$, $\Gamma_1 = D$ and then recursively for k = 2, 3, … also define

(16a) $M_k = (I - A\Omega_{k-1})^{-1} M_{k-1}$

(16b) $\Omega_k = (I - A\Omega_{k-1})^{-1} C$

(16c) $\Gamma_k = (I - A\Omega_{k-1})^{-1}(D + A\Gamma_{k-1}R)$.

These expressions clearly define a unique iterative process. Provided that for all k = 1, 2, … the regularity condition $\det(I - A\Omega_k)^{-1} \neq 0$ is satisfied, then also

(17) $x_t = M_k E_t x_{t+k} + \Omega_k x_{t-1} + \Gamma_k z_t$

for all k = 1, 2, …. Then the issue is whether the matrices M_k, Ω_k, and Γ_k converge as $k \rightarrow \infty$. If so, a "forward convergence condition" (FCC) is satisfied. If in addition $M_k \rightarrow O$ the limiting case of the relationship is

(18) $x_t = \Omega_\infty x_{t-1} + \Gamma_\infty z_t$,

which defines a single RE solution that excludes extraneous "bubble" components. This "forward method" (FM) solution then provides a unique solution in terms of fundamentals, which Cho and Moreno recommend as the most attractive solution among those that do not involve "bubbles." The clarity of the uniqueness result is quite attractive.[16] One limitation of this approach is that the discussion is restricted to solutions based only on "fundamentals"; no consideration of "sunspot" solutions is provided.

[16] The iterative solution is similar to ones discussed by Flood and Garber (1980) and Sargent (1979, pp. 192-5) but the Cho and Moreno analysis provides generality and uniqueness results that seem to be quite original.

6. Alternative Selection Criteria: McCallum

In a very recent working paper (McCallum, 2011b), I have described a plausibility condition that is conceptually distinct from the LS learnability condition promoted in Sections 1-4 above. It is that the solution coefficients included in the matrices Ω and Γ of equation (11) do not feature infinite discontinuities in the limit as the parameter matrix A→O. The idea in this case is that very small changes, in those structural parameters that reflect the influence of expectational variables on agents' supply-demand choices, should not result in enormous changes in the resulting choices. In my working paper it is shown that in the general linear model (9) (10) this condition is satisfied by a single solution.[17] It transpires, moreover, that this single solution is the same as the "minimum state variable" solution originally recommended in McCallum (1983). A proof is provided below in Appendix B and the criterion is given a "causality" interpretation in McCallum (2011a).

Cho and Moreno (2011, p. 269) have noted explicitly that their forward-method solution does not always coincide with the MSV solution, that is, the MSV/continuity criterion does not always agree with the forward-solution criterion. In this regard Cho and Moreno provide an example (2011, p. 269) in which there is a MSV solution but the forward solution does not exist. However, in this example the MSV solution does not have the property of LS learnability. Accordingly, since my analysis views both continuity/MSV and learnability as necessary conditions for a solution to be plausible, there is apparently no actual disagreement provided by this particular example.

7. The Role of Money

The model (1) – (3) is often interpreted as pertaining to a "cashless economy" in which

[17] This result pertains to the case in which sunspot solutions are permitted, as well as to the case in which only fundamental solutions are considered.

there is no medium of exchange, i.e., no money. That is not my interpretation, however; instead I would take this to be a model in which there is a medium of exchange (MOE) and a resulting money demand function of the form[18]

$$(19)\qquad m_t - p_t = \gamma_0 + \gamma_1 y_t + \gamma_2 R_t + \zeta_t$$

where $\gamma_1 > 0, \gamma_2 < 0,$ and the disturbance ζ_t is presumably related to the other shock processes in the model (1)-(3). Then IF the central bank conducts policy by choosing R_t as specified by (3), this relation (19) will serve only to specify how much money the central bank has to supply each period in order to implement its policy as specified in (3); relation (19) will have no effect on the behavior of either y_t or p_t and does not need to be considered at all.

It is true, of course, that a money demand function of the form in (19) is a special case that will come about only if the way in which money affects transaction costs in the (implicit) model that underlies (1)-(2) involves a transaction-cost function that is *separable*. And that is a special and quite unlikely form for the transaction cost function to assume. But my own attempt to estimate the magnitude of the effect on the model's properties of specifying a more realistic transaction cost function led to the conclusion that the quantitative effects of this correction are negligible.[19]

Accordingly, the standard analytical approach of the New Keynesian[20] mainstream of recent years does not seem to be fundamentally flawed, in the sense that it is applicable to an economy in which there is in fact a tangible medium of exchange. Also, it is in my opinion

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[18] Here m_t denotes the relevant money stock, expressed in fractional deviation units.
[19] See McCallum (2001). A similar exercise was independently conducted by Woodford (2003, pp. 111-123) with results that were extremely close to mine. Ireland (2004) took a different approach but obtained similar conclusions.
[20] The term "New Keynesian," when applied to the mainstream analytical approach of the past 20 years, is in my opinion a misnomer. This approach seems closer to the "monetarist" position of Friedman, Schwartz, Brunner, Meltzer, Laidler, and Parkin during the Keynesian vs. Monetarist debates of the 1970s than to the "Keynesian" position of Tobin, Modigliani, Samuelson, Solow, Gordon, Okun, and Klein.

appropriate that this analysis includes a price adjustment relationship (aka "Phillips curve") that involves some sluggishness in prices, thereby imparting a non-trivial effect of monetary policy on the cyclical behavior of real output and employment.[21] Whether the details of the usual Calvo-type adjustment relationship are well enough understood to enable central banks to successfully conduct activist countercyclical policy in a desirable manner is less clear. It may well be that the best thing that central banks can do to stimulate output and employment is to keep inflation low and steady. No analysis of these issues has been attempted here.

8. Conclusion

A brief summary of this paper's argument is as follows. In recent monetary policy analysis, it has been common practice to view models as possessing *determinacy* if they feature a single RE solution that is dynamically stable. Cochrane (2007) has emphasized that the single-stable- solution (SSS) condition is not sufficient as a criterion of determinacy, however, because in typical New Keynesian models there exists a dynamically explosive solution that is not ruled out by any transversality condition and accordingly can be eliminated only by an arbitrary dictum. McCallum (2009b) agrees with this specific proposition, but shows that in these models it is typically the case that the explosive solutions in question are not least-squares learnable. Further, he argues that such learnability should be considered a necessary condition for a solution to be regarded as a model's prediction of the depicted economy's behavior since it amounts to a *feasibility* condition that pertains to quantitative information available to individual agents. Consequently, he argues that, despite Cochrane's point, the solution typically utilized in recent policy analysis is in many (but not all) cases the appropriate one. Cochrane's (2009) response contends that there are three weaknesses in McCallum's argument. Here we elaborate

[21] To exclude any such relationship would be to imply that an extreme tight money episode engineered by the central bank would not induce a recession—suggesting that the Volcker Disinflation was just an accident.

on McCallum's (2009c) claims that in all three cases Cochrane's argument is analytically incorrect or inapplicable. First, the presence of unobserved exogenous shocks does not overturn learnability conclusions. Second, Cochrane's argument about "hyperinflationary threats" is not consistent with the analytical setting in which the argument is normally conducted, namely, one in which the central bank is following a specified policy rule. Third, the point that a particular structural parameter, concerning the central bank's policy behavior, is not identifiable by an econometrician studying the economy-plus-policy process is not relevant to the learning process for the private-sector agents in the model. Their learning concerns forecasting of inflation and output in the model economy from a reduced form perspective; the identification of a structural parameter by these agents is not necessary for this step.

In addition, the paper reviews two recent proposals for selection of a single RE solution as relevant for policy analysis. First, Cho and Moreno (2011) have promoted a recursive "forward solution" yielding a fundamentals solution that includes no "bubble" components; they show that such a solution is unique in all cases in which it exists. Second, McCallum (2011b) has shown that in all linear models of a standard broad class there is a single RE solution that is plausible in the sense of not implying infinite discontinuities in solution parameters (and impulse response functions) for extremely small changes in structural parameters pertaining to the magnitude of response of supply-demand actions to expectational variables.

Finally, the paper considers the question of whether a policy rule that involves period-by-period control of a nominal interest rate is necessarily unsatisfactory in some fundamental manner, presuming that the economy is one in which a tangible, transaction-facilitating, medium of exchange exists. The paper's conclusion is that the answer is "no," which is not the same as arguing that an interest rate instrument is inherently superior to a monetary-aggregate instrument such as the monetary base.

Appendix A

It may be useful, for expository purposes, to describe the LS learning process again, but now for the special model of equation (4), which we here write as $\pi_t = aE_t\pi_{t+1} + u_t$, where $a = 1/(1+\mu_1)$ and $u_t = -ae_t$. The RE solution corresponding to (6) is then $\pi_t = \psi_1 u_t$ with $\psi_1 = \frac{1}{1-a\rho}$. For the LS learning process we assume that agents do not know the values of a or ψ_1 and accordingly use in place of $E_t\pi_{t+1}$ the value π^e_{t+1} to be defined momentarily. Thus in each period the agents estimate the relationship $\pi_t = \psi_{1t}u_t$, where the estimate ψ_{1t} is obtained by least squares regression with data from all previous periods: $\psi_{1t} = [\sum_{\tau=1}^{t-1} u_\tau^2]^{-1} \sum_{\tau=1}^{t-1} u_\tau \pi_\tau$.[22] Then expectations are given by $\pi^e_{t+1} = \psi_{1t}\rho u_t$ and π_t is generated as

(A-1) $$\pi_t = a(\psi_{1t}\rho u_t) + u_t.$$

In this simple setting, it is easy to see that if the process is such that $\psi_{1t} \to \psi_1 = \frac{1}{1-a\rho}$, then in the limit we have

(A-2) $$\pi_t = a\left[\left(\frac{1}{1-a\rho}\right)\rho u_t\right] + u_t = \left[\frac{a\rho}{1-a\rho} + 1\right]u_t = \left[\frac{1}{1-a\rho}\right]u_t,$$

that is, the RE solution (6). Does the process converge? Clearly the learnability conditions analogous to (14a)-(14c) are that the 1×1 matrices a, 0, and aρ all have eigenvalues with real parts less than 1, that is, that $a < 1$ and $a\rho < 1$. But $a < 1$ is implied by the Taylor Principle condition $\mu_1 > 0$ whereas $|\rho| < 1$ by assumption. Consequently, we see that learnability prevails

[22] In principle, a constant term should also be included. It is omitted here for expositional simplicity.

for the MSV solution (6). In addition, the non-learnability result for the alternative solution (8) can be obtained in an analogous fashion.

Appendix B

The object here is to show that in the model (9)(10) the solution (11) for which $\Omega \to C$ as $A \to O$ is the same solution as the one for which $\Omega \to O$ as $C \to O$. Let us begin with the case in which A is nonsingular. Then we can express the crucial matrix quadratic (12a) as

(B-1) $$\begin{bmatrix} \Omega^2 \\ \Omega \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}C \\ I & O \end{bmatrix} \begin{bmatrix} \Omega \\ I \end{bmatrix}.$$

Let M denote the square matrix of order 2m × 2m.[23] Clearly its eigenvalues are the numbers denoted λ that satisfy

(B-2) $$\det(M - \lambda I) = \det \begin{bmatrix} A^{-1} - \lambda I & -A^{-1}C \\ I & O - \lambda I \end{bmatrix} = 0.$$

Next, an identity for partitioned matrices reported by Johnston (1972, eqn. 4-37, p. 95) implies that

(B-3) $$\det(M - \lambda I) = \det(A^{-1} - \lambda I)\, \det[-\lambda I + (A^{-1} - \lambda I)^{-1} A^{-1} C] = 0.$$

Thus we see, from the latter, that half of the eigenvalues of M are the eigenvalues of A^{-1}, while the other half depend upon both A and C. Then by further inspection of (B-3) we see that when C = O, the second half of the λs are all equal to 0. Thus the single solution given by the particular eigenvalue arrangement for which the eigenvalues of Ω all approach zeros as $C \to O$, simultaneously has the other half of the eigenvalues of M approaching the eigenvalues of A^{-1}.

. Now consider the same arrangement but with C held fixed and consider the implication of $A \to O$. Then the eigenvalues that approached zeros before now approach the eigenvalues of

[23] Note that here the matrix M is not related to the matrices denoted M_k in section 5.

C while the eigenvalues that approached those of A^{-1} before now approach $\pm \infty$. This establishes the result at issue for the case in which A is nonsingular.

When instead A is singular, similar results obtain but with the matrix A being replaced in the argument by the matrix $F = (I - A\Omega)^{-1}A$.[24] The system eigenvalues then include those of F^{-1} and Ω, instead of those of A^{-1} and Ω. As $C \rightarrow O$, we have the m eigenvalues of Ω approaching zeros and the other m eigenvalues approaching those of F^{-1}. So with the same arrangement, we find that as $A \rightarrow O$, the eigenvalues of F^{-1} each approach $\pm \infty$.

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[24] Again, see McCallum (2007, pp. 1381-1383) for this result.

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