

# Financial Networks, Cross Holdings, and Limited Liability

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I discuss a network of banks which are linked with each other by financial obligations and cross holdings. Given an initial endowment the value of the obligations and the equity values of the banks are determined endogenously in a way consistent with priority of debt and limited liability of equity. Even though neither equity values nor debt values are necessarily unique the value of debt and equity holdings of outside investors is uniquely determined. An algorithm to calculate debt and equity values is developed.

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# 1. Introduction

Banks are connected with each other through a widely ramified network of financial claims and obligations. The value of these claims depends on the financial health of the obligor who himself might be an obligee such that his financial health depends on his obligors. Through these linkages financial distress of one bank might draw other banks into default and thereby create a domino effect of bank failures – a systemic crisis. Under the labels of *systemic risk* and *contagion* this problem has gained considerable attention in the literature on financial intermediaries (among others Allen and Gale (2000), Giesecke and Weber (2004), Rochet and Tirole (1997), and Shin (2008)).

According to Boyd et al. (2005) the social costs of a systemic crisis range from 60% to 300% of GDP. The prevention of a breakdown of financial intermediation is of vital interest for central banks and regulatory authorities. To assess the probability and severity of such a cascade of defaults caused by interbank lending a large number of central banks perform counterfactual simulations.<sup>1</sup> In a first step bilateral credit exposures in the interbank market are determined.<sup>2</sup> Given these exposures there are two approaches in the literature to simulate contagion. The first approach introduced by Furfine (2003) assumes that a particular bank is not able to honor its obligations. All creditors of this bank lose an exogenously specified fraction of their claims against this bank (loss given default). If the losses of an affected bank exceed its capital, the bank is in default, too. In the next round the creditors of all defaulting banks lose a fraction of their claims against these banks. Again these losses are compared to the capital available. The procedure is iterated until no additional bank goes bankrupt. If this simulation is performed for each bank, it allows to determine which banks are systemically relevant in the sense that they trigger contagious defaults. Yet, the approach is not able to assess the probability that a particular bank defaults. Moreover, loss given default is not endogenous but exogenously given.

The second simulation approach is based on a model developed by Eisenberg and Noe (2001). In their framework banks are not only linked with each other via the interbank market but are also endowed with exogenous income. Under the assumption of limited liability of equity and absolute priority of debt Eisenberg and Noe (2001) show that for a given level of exogenous income the equity values of the banks, default and loss given default can be determined endogenously. Elsinger et al. (2006a) introduced this approach to the contagion literature by applying it to the Austrian banking system.

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<sup>1</sup>The different approaches are reviewed in Upper (2007).

<sup>2</sup>Typically these exposures are not readily available. They have to be estimated from aggregate data. Mistrulli (2006) discusses the consequences of estimation errors.

They take the net position of all non-interbank related parts of the balance sheet as exogenous income. Using standard risk management techniques they simulate changes in the value of this exogenous income for all banks in the system simultaneously. Given such a scenario the interbank market is cleared and the equity values of the banks and the values of the interbank debt contracts are determined. The advantage of this approach is that it not only allows to identify systemically important banks but also allows to assess the probability of default and the level of loss given default endogenously. Yet, two important features commonly observed in real world networks are not included in this framework. Firstly, cross shareholdings between the banks are not explicitly modeled. The value of such holdings has to be treated as a part of exogenous income. This leads to inconsistencies. The simulated value of the holding as a part of exogenous income will typically not equal the value after the clearing procedure. Secondly, the seniority structure of debt is not modeled. It is assumed that all debt in the interbank market is of the same seniority and that it is junior relative to all other debt claims against the banks.

The aim of this paper is to extend the work of Eisenberg and Noe (2001) by taking cross holdings and a detailed seniority structure of debt explicitly into account. Banks are modeled as nodes in a network which are endowed with exogenous income. They may have nominal obligations to other nodes in the network and to outside creditors. Furthermore the nodes may hold equity shares of other nodes. I assume limited liability, absolute priority of debt, and proportional rationing of debt claims in case of default. Additionally, there must not exist a subset of nodes where each node in the subset is owned entirely by the other nodes in the subset. Given these assumptions neither equity nor debt values are necessarily unique. Yet, banks with non-unique equity values have to be entirely owned either by other banks with non-unique equity values or zero equity value. If the debt payments of a bank are non-unique it has to hold that all claimants are themselves banks with either non-unique debt payments or non-unique equity value. Or to put it the other way round, the values of debt and equity claims that are held by outside investors are unique. Moreover, given there are no bankruptcy costs it never pays for outside investors to bail out defaulting banks.

The paper is organized as follows. In Section 2 a financial network with cross holdings is presented and the main concepts are developed. In Section 3 I prove that a clearing vector exists and characterize networks for which the clearing vector is unique. In Section 4 a solution algorithm is presented which lacks the elegance of the fictitious default algorithm developed by Eisenberg and Noe (2001) but which is applicable under weaker assumptions on the structure of the network. This algorithm allows to incorporate cross

holdings and a detailed seniority structure. I discuss the comparative statics in Section 5. In a financial network without cross holdings Eisenberg and Noe (2001) show that equity values are convex and debt values are concave in the exogenous income. Using a simple example I show that is not true as soon as cross holdings are included. In Section 6 the model is augmented with a detailed seniority structure. It is shown that the main results remain valid. Section 7 concludes the paper.

## 2. The Model

Consider an economy populated by  $n$  banks constituting a *financial network*. Each of these banks is endowed with an exogenous income  $e_i \in \mathbb{R}$  which may be negative. Exogenous income may be interpreted as a random variable. For each draw of  $e = (e_1, \dots, e_n)'$  the system is cleared. All the results derived are conditional on a particular draw.

Without a detailed priority structure of debt,  $e_i$  may be interpreted as operating income minus all liabilities except the most junior. Ruling out that  $e_i$  might be negative would be equivalent to assuming that all liabilities except the most junior are always repaid in full.<sup>3</sup> Any bank may hold shares of companies outside the network. The value of these holdings is not determined endogenously. It is included in  $e$ .

Banks may have nominal obligations to other banks in the network. The structure of these liabilities is represented by an  $n \times n$  matrix  $L$  where  $L_{ij}$  represents the nominal obligation of bank  $i$  to bank  $j$ . These liabilities are nonnegative and the diagonal elements of  $L$  are zero as banks are not allowed to hold liabilities against themselves. Liabilities to creditors outside the network are denoted by  $D_i \geq 0$ . Furthermore, banks may hold shares of other banks which are denoted by the matrix  $\Theta \in [0, 1]^{n \times n}$  where  $\Theta_{ij}$  is the share of bank  $i$  held by bank  $j$ . A bank may hold its own shares ( $\Theta_{ii} > 0$ ).

Suppose there are two banks,  $A$  and  $B$ . Bank  $A$  has an exogenous income of 1, no outstanding debt, and owns bank  $B$  entirely. On the other hand bank  $B$  has an exogenous income of 2, no debt and owns bank  $A$ . The equity value of  $A$  equals 1 plus the equity value of  $B$ .  $B$ 's equity value equals 2 plus the equity value of  $A$ . The only solution would be that both banks have an equity value of infinity. In this example ownership is not well defined irrespective whether there is limited or unlimited liability. To avoid this problem it suffices to assume that there is no group of banks where *each* bank within the group is completely owned by other banks in that group. In particular, a bank must not own itself entirely ( $\Theta_{ii} < 1$ ).

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<sup>3</sup>In Section 6 the framework is extended to deal with different seniority classes.

**Assumption 1.** *There exists no subset  $\mathcal{I} \subset \{1, \dots, n\}$  such that*

$$\sum_{j \in \mathcal{I}} \Theta_{ij} = 1 \quad \text{for all } i \in \mathcal{I}.$$

*with  $\Theta \in [0, 1]^{n \times n}$  and  $\Theta \vec{1} \leq \vec{1}$  where  $\vec{1}$  is an  $n \times 1$  vector of ones.  $\Theta$  is called a holding matrix if it fulfills this assumption.*

A bank is defined to be in default whenever exogenous income plus the amounts received from other nodes plus the value of the holdings are insufficient to cover the bank's nominal liabilities.<sup>4</sup> Throughout the paper I assume that bank defaults do not change the prices outside of the network, i.e.  $e$  is independent of defaults and exogenous. If default changes prices due to e.g. fire sales, the story is different. Only in the special case where defaults unequivocally decrease  $e$  the main results still hold.<sup>5</sup>

In case of default the clearing procedure has to respect three criteria:

1. limited liability: which requires that the total payments made by a node must never exceed the sum of exogenous income, payments received from other nodes, and the value of the holdings,
2. priority of debt claims: which requires that stockholders receive nothing unless the bank is able to pay off all of its outstanding debt completely, and
3. proportionality: which requires that in case of default all claimant nodes are paid off in proportion to the size of their claims on firm assets.

To operationalize proportionality let  $\bar{p}_i$  be the total nominal obligations of node  $i$ , i.e.

$$\bar{p}_i = \sum_{j=1}^n L_{ij} + D_i$$

and define the proportionality matrix  $\Pi$  by

$$\Pi_{ij} = \begin{cases} \frac{L_{ij}}{\bar{p}_i} & \text{if } \bar{p}_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

Evidently, it has to hold that  $\Pi \cdot \vec{1} \leq \vec{1}$ .

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<sup>4</sup>A bank is in default if liabilities exceed assets. Using a violation of capital requirements as default threshold does not change the main results.

<sup>5</sup>Shin (2008) endogenized  $e$  in a model without cross share holdings.

To simplify notation I define for any two (column) vectors  $x, y \in \mathbb{R}^n$  the lattice operations

$$\begin{aligned} x \wedge y &:= (\min(x_1, y_1), \dots, \min(x_n, y_n))' \\ x \vee y &:= (\max(x_1, y_1), \dots, \max(x_n, y_n))'. \end{aligned}$$

Let  $p = (p_1, \dots, p_n)' \in \mathbb{R}_+^n$  be an arbitrary vector of payments made by banks to their interbank and non interbank creditors. To define the equity values  $V$  of the banks assume for a moment that these values are exogenously given ( $V \geq \vec{0}$ ) and define the map

$$\Psi^1(V, p, e, \Pi, \Theta) = [e + \Pi'p - p + \Theta'V] \vee \vec{0} \quad (1)$$

where  $\vec{0}$  denotes the  $n \times 1$  dimensional zero vector.  $\Psi^1$  returns the values of the nodes given  $V$  and  $p$ . A necessary condition for  $V$  to be a vector of equity values is that  $V$  is a fixed point,  $V^*(p)$ , of  $\Psi^1(\cdot; p, e, \Pi, \Theta) : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ , i.e.

$$V^*(p) = [e + \Pi'p - p + \Theta'V^*(p)] \vee \vec{0}. \quad (2)$$

If  $\Theta$  is a holding matrix, Lemma 4 in the Appendix establishes that  $V^*(p)$  is unique for any  $p$ . However, for arbitrary  $p$  it is possible that the equity value of bank  $i$  is positive ( $V_i^*(p) > 0$ ) but the actual payments made do not cover the liabilities ( $p_i < \bar{p}_i$ ). In this case absolute priority would not hold. Given  $p$  the amount available for bank  $i$  to pay off its debt equals  $e_i + \sum_{j=1}^n \Pi_{ji}p_j + \sum_{j=1}^n \Theta_{ji}V_j^*(p)$ . If this amount is less than zero, bank  $i$  has to pay nothing due to limited liability. If this amount is larger than the liabilities ( $\bar{p}_i$ ), bank  $i$  must pay off its debt completely because of absolute priority. If the amount available is in the range from zero to  $\bar{p}_i$ , it is distributed proportionally amongst the debt holders again due to absolute priority. A vector of payments  $p^*$  respects the clearing criteria if

$$p_i^* = \begin{cases} 0 & \text{for } e_i + \sum_{j=1}^n (\Pi_{ji}p_j^* + \Theta_{ji}V_j^*(p^*)) \leq 0 \\ e_i + \sum_{j=1}^n (\Pi_{ji}p_j^* + \Theta_{ji}V_j^*(p^*)) & \text{for } 0 \leq e_i + \sum_{j=1}^n (\Pi_{ji}p_j^* + \Theta_{ji}V_j^*(p^*)) \leq \bar{p}_i \\ \bar{p}_i & \text{for } \bar{p}_i \leq e_i + \sum_{j=1}^n (\Pi_{ji}p_j^* + \Theta_{ji}V_j^*(p^*)). \end{cases}$$

For such a payment vector  $V_i^*(p^*) > 0$  implies that  $p_i^* = \bar{p}_i$ .  $V^*(p^*)$  is a vector of equity values consistent with limited liability, absolute priority, and proportional rationing in the case of default.

**Definition 1.** A vector  $p^* \in [\vec{0}, \bar{p}]$  is a **clearing payment vector** if

$$p^* = \left\{ [e + \Pi'p^* + \Theta'V^*(p^*)] \vee \vec{0} \right\} \wedge \bar{p} \quad (3)$$

where  $V^*(p^*)$  is the unique fixed point of  $\Psi^1(\cdot; p^*, e, \Pi, \Theta)$ .

Alternatively, a clearing vector  $p^*$  can be characterized as a fixed point of the map  $\Phi^1(\cdot; \Pi, \bar{p}, e, \Theta) : [\vec{0}, \bar{p}] \rightarrow [\vec{0}, \bar{p}]$  defined by

$$\Phi^1(p; \Pi, \bar{p}, e, \Theta) = \left\{ [e + \Pi'p + \Theta'V^*(p)] \vee \vec{0} \right\} \wedge \bar{p} \quad (4)$$

$\Phi^1$  returns the minimum of the maximum possible payment and the promised payment  $\bar{p}$ . Hence, any  $p \geq \Phi^1(p)$  is compatible with absolute priority but not necessarily with limited liability. In the framework without cross holdings ( $\Theta = \mathbf{0}_{n,n}$ ) Eisenberg and Noe (2001) show that such a clearing vector exists. Furthermore, they are able to specify sufficient conditions to guarantee uniqueness.

### 3. Existence and Uniqueness of a Clearing Payment Vector

To prove the existence of a clearing vector Eisenberg and Noe (2001) employ the Tarski fixed point theorem (Theorem 11.E in Zeidler (1986)). As  $V^*(p)$  and thereby  $\Phi^1$  are not monotone in  $p$  the Tarski fixed point theorem can not be applied forthrightly. The problem has to be rephrased. Suppose – for a moment – that  $\Theta = \mathbf{0}_{n,n}$  and that a clearing vector  $p^*$  exists. The value of bank  $i$  is given by  $V_i^*(p^*) = \max(e_i + \sum_{j=1}^n \Pi_{ji}p_j^* - p_i^*, 0)$ .  $V_i^*(p^*) > 0$  implies that  $p_i^* = \bar{p}_i$ . The value of bank  $i$  may therefore be written as  $W_i^*(p^*) = e_i + \sum_{j=1}^n \Pi_{ji}p_j^* - \bar{p}_i$ . If  $\Theta$  is arbitrary, this translates into

$$W^*(p^*) = [e + \Pi'p^* - \bar{p}] + \Theta'(W^*(p^*) \vee \vec{0}). \quad (5)$$

$W^*(p^*)$  is the vector of node values under the assumption of limited liability for the cross holdings.  $W^*(p^*)$  is not necessarily nonnegative. But  $(W^*(p^*) \vee \vec{0})$  equals the equity values  $V^*(p^*)$  as will be proved in Theorem 1.

More formally,  $W^*(p)$  may be defined as a fixed point of the function of  $\Psi^2(\cdot; p, \bar{p}, e, \Pi, \Theta) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , given by

$$\Psi^2(W, p, \bar{p}, e, \Pi, \Theta) = e + \Pi'p - \bar{p} + \Theta'(W \vee \vec{0}). \quad (6)$$

Lemma 5 in the Appendix establishes that  $W^*(p)$  is unique for arbitrary  $p$  and more importantly that  $W^*(p)$  is increasing in  $p$ . The definition of a clearing vector has to be adjusted to this alternative definition of equity values by substituting  $W^*(p) \vee \vec{0}$  for

$V^*(p)$ . Define the map  $\Phi^2(p; \Pi, \bar{p}, e, \Theta) : [\vec{0}, \bar{p}] \rightarrow [\vec{0}, \bar{p}]$  by

$$\Phi^2(p; \Pi, \bar{p}, e, \Theta) = \left\{ \left[ e + \Pi'p + \Theta'(W^*(p) \vee \vec{0}) \right] \vee \vec{0} \right\} \wedge \bar{p} = \left\{ [W^*(p) + \bar{p}] \vee \vec{0} \right\} \wedge \bar{p} \quad (7)$$

where  $W^*(p)$  is the unique fixed point of  $\Psi^2$ . If  $p^*$  is a fixed point of  $\Phi^2(p)$  then it holds that  $W_i^*(p^*) \geq 0$  is equivalent to  $p_i^* = \bar{p}_i$ . So, we may call  $p^*$  a clearing vector and  $W^*(p^*) \vee \vec{0}$  the vector of equity values.

$V^*(p)$  is not equal to  $(W^*(p) \vee \vec{0})$  for arbitrary  $p$ . But if  $p$  is a supersolution of  $\Phi^1(p)$  or  $\Phi^2(p)$ , i.e.  $p \geq \Phi^1(p)$  or  $p \geq \Phi^2(p)$ , then  $V^*(p) = (W^*(p) \vee \vec{0})$ .

**Theorem 1.** *Let  $\hat{p} \in [\vec{0}, \bar{p}]$  be a (super)solution of  $\Phi^1(p)$ , i.e.  $\hat{p} \geq \Phi^1(\hat{p}; \Pi, \bar{p}, e, \Theta)$ . Then  $\hat{p}$  is a (super)solution of  $\Phi^2(p)$  with  $W^*(\hat{p}) = e + \Pi'\hat{p} - \bar{p} + \Theta'V^*(\hat{p})$ . If  $\hat{p} \in [\vec{0}, \bar{p}]$  is a (super)solution of  $\Phi^2(p)$  then  $\hat{p}$  is a (super)solution of  $\Phi^1(p)$  with  $V^*(\hat{p}) = (W^*(\hat{p}) \vee \vec{0})$ .*

As a consequence any fixed point of  $\Phi^2(p)$  is a fixed point of  $\Phi^1(p)$  and vice versa. To prove that a clearing vector exists it suffices to show that  $\Phi^2(p)$  has a fixed point. By construction  $\Phi^2(\vec{0}) \geq \vec{0}$  and  $\Phi^2(\bar{p}) \leq \bar{p}$ . The Tarski fixed point theorem guarantees that there exists a least and a greatest fixed point for  $\Phi^2(p)$  if  $\Phi^2(p)$  is a monotone increasing function on the complete lattice  $[0, \bar{p}]$ . Lemma 5 in the Appendix shows that  $W^*(p)$  and thereby  $\Phi^2(p)$  are increasing in  $p$ .

**Theorem 2.** *There exists a greatest ( $p^+$ ) and a least ( $p^-$ ) clearing vector.*

If the clearing vector is not unique it might happen that the equity values of the banks are different for different clearing vectors. In particular it could happen that a bank that is in default at  $p^-$  might have a positive equity value at  $p^+$ . Eisenberg and Noe (2001) show that for  $\Theta = \mathbf{0}_{n,n}$  the equity values of the nodes do not depend on the chosen clearing vector. A bank defaulting at  $p^-$  might not default at  $p^+$ . Yet, the equity value at  $p^+$  has to be zero and the bank is only just solvent. In the more general framework under consideration in this paper the situation is more complicated as is illustrated by the following example.

**Example 1.** *Assume that the network is characterized by the following parameters.*

$$e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

*It is easy to check that any  $p(\lambda) = (1, \lambda)'$  for  $\lambda \in [0, 1]$  is a clearing vector with corresponding equity value  $V^*(\lambda) = (\lambda, 0)'$ . The equity value of bank 1 is not unique. To see why note that any amount  $\lambda \in [0, 1]$  paid by bank 2 is received entirely by bank 1 as it*



is the only obligee of bank 2. Bank 1 is able to cover all liabilities by exogenous income. The additional payment of  $\lambda$  increases the equity value by the very same amount. But bank 1 is entirely owned by bank 2. Hence, the value of the holdings of bank 2 increases by  $\lambda$  and the initial payment of  $\lambda$  is affordable.

If  $\Pi_{21}$  or  $\Theta_{12}$  or both were smaller than 1, the unique clearing payment would be  $(1, 0)'$  with corresponding equity values  $(0, 0)'$ . Moreover, if the aggregate exogenous income were not equal to 1, the clearing vector would be unique, too.

The existence of multiple clearing vectors in the example is driven by two facts. First there exists a subset of banks with the property that each bank within this subset is either entirely owned by other banks in the subset or all creditors of this bank belong to the subset. Bank 1 is owned by bank 2 and bank 1 is the only creditor of bank 2. Additionally, the aggregate exogenous income of all banks in this particular subset equals exactly the amount owed to creditors outside the subset. Aggregate exogenous income of the two banks in the example equals 1. Bank 1 owes 1 unit to creditors outside the network.

The next theorem establishes that exactly these two properties are necessary for the existence of multiple clearing vectors.

**Theorem 3.** *Suppose the network allows for multiple clearing vectors  $p^+ \neq p^-$  with corresponding equity values  $V^+$  and  $V^-$ . Let  $\mathcal{I}^0$  be the subset of banks with non-unique equity values and  $\mathcal{I}^1$  be the subset of banks with non-unique clearing payments. Denote the union of  $\mathcal{I}^0$  and  $\mathcal{I}^1$  by  $\mathcal{I}$ . It has to hold that*

1. *all banks in  $\mathcal{I}^0$  are entirely owned by banks in  $\mathcal{I}$ , i.e.  $\sum_{j \in \mathcal{I}} \Theta_{ij} = 1$  for all  $i \in \mathcal{I}^0$ ,*
2. *the only creditors of banks in  $\mathcal{I}^1$  are banks in  $\mathcal{I}$ , i.e.  $\sum_{j \in \mathcal{I}} \Pi_{ij} = 1$  for all  $i \in \mathcal{I}^1$ , and*
3. *the sum of the obligations of banks in  $\mathcal{I}$  to banks not in  $\mathcal{I}$  has to equal the aggregate exogenous income of banks in  $\mathcal{I}$  plus the value of all claims of banks in  $\mathcal{I}$  against banks not in  $\mathcal{I}$ , i.e.*

$$\sum_{i \in \mathcal{I}} \left( 1 - \sum_{j \in \mathcal{I}} \Pi_{ij} \right) \bar{p}_i = \sum_{i \in \mathcal{I}} e_i + \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Pi_{ij} \right) p_i^* + \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Theta_{ij} \right) V_i^*$$

where  $p_i^* = p_i^+ = p_i^-$  and  $V_i^* = V_i^+ = V_i^-$  for  $i \notin \mathcal{I}$ .

The theorem shows that the clearing vector is necessarily unique if outside investors hold some debt and equity claims against each bank or if  $e$  is sufficiently large. If

aggregate exogenous income of banks in  $\mathcal{I}$  exceeds the net obligations to banks outside of  $\mathcal{I}$  the clearing vector has to be unique.

**Corollary 1.** *Suppose for any subset  $\mathcal{I}$  of banks with*

$$\sum_{j \in \mathcal{I}} \Theta_{ij} = 1 \quad \text{or} \quad \sum_{j \in \mathcal{I}} \Pi_{ij} = 1 \quad \text{for all } i \in \mathcal{I}$$

*it holds that  $\sum_{i \in \mathcal{I}} e_i > \sum_{i \in \mathcal{I}} (1 - \sum_{j \in \mathcal{I}} \Pi_{ij}) \bar{p}_i$  then the clearing vector is unique. For  $\Theta = \mathbf{0}_{n,n}$  the clearing vector is unique if  $\sum_{i \in \mathcal{I}} e_i > 0$ .*

*Proof.* Only the claim for  $\Theta = \mathbf{0}_{n,n}$  remains to be shown. In this case  $\sum_{j \in \mathcal{I}} \Pi_{ij} = 1$  for all  $i \in \mathcal{I}$  and  $\sum_{i \in \mathcal{I}} (1 - \sum_{j \in \mathcal{I}} \Pi_{ij}) \bar{p}_i = 0$ .  $\square$

Outside investors hold a share of  $1 - \sum_{j=1}^n \Theta_{ij}$  of bank  $i$ 's equity and a share of  $1 - \sum_{j=1}^n \Pi_{ij}$  of bank  $i$ 's debt. If the equity value of bank  $i$  is non-unique then the bank is entirely owned by other banks, i.e.  $1 - \sum_{j=1}^n \Theta_{ij} = 0$ . Analogously, if the clearing payment of bank  $i$  is non-unique then the only obligees are other banks, i.e.  $1 - \sum_{j=1}^n \Pi_{ij} = 0$ . As a consequence outside investors wealth does not depend on the chosen clearing vector.

**Corollary 2.** *The value of an outside investor's portfolio is independent of the chosen clearing vector. In particular, for clearing vectors  $p^+$  and  $p^-$  and corresponding equity values  $V^+$  and  $V^-$  it holds that  $\vec{1}'(\mathbf{I} - \Theta')(V^+ - V^-) = \vec{1}'(\mathbf{I} - \Pi')(p^+ - p^-) = 0$ .*

To model bankruptcy costs, the framework needs to be adapted. Denote the vector of bankruptcy costs by  $b(p)$  and assume that  $b(p)$  is decreasing in  $p$ . The simplest type of bankruptcy costs would be such that the defaulting bank  $i$  loses a specified exogenous amount  $c_i > 0$ , i.e.  $b_i(\bar{p}_i) = 0$  and  $b_i(p_i) = c_i$  for all  $p_i < \bar{p}_i$ . A sufficient condition for a clearing vector to exist is that  $W^*(p)$  is increasing in  $p$ . With bankruptcy costs  $W^*(p)$  has to be defined as a fixed point of

$$\hat{\Psi}^2(W, p, \bar{p}, e, \Pi, \Theta) = e(p) + \Pi'p - \bar{p} + \Theta'(W \vee \vec{0}) \quad (8)$$

where  $e(p) = e - b(p)$ .  $W^*(p)$  is unique and increasing in  $p$  by Lemma 5 in the Appendix. Applying the Tarski fixed point theorem to

$$\hat{\Phi}^2(p; \Pi, \bar{p}, e, \Theta) = \left\{ \left[ e(p) + \Pi'p + \Theta'(W^*(p) \vee \vec{0}) \right] \vee \vec{0} \right\} \wedge \bar{p} \quad (9)$$

yields that a greatest and a least clearing vector exist. If  $e$  depends on  $p$ , the conditions stated in Theorems 3 and 1 do not suffice to guarantee a unique clearing vector. Given

that bankruptcy is costly different clearing vectors induce different equity values. As a consequence, the value of an outside investor's portfolio depends on the chosen clearing vector. Additionally, it may be profitable to bail out defaulting banks to save the bankruptcy costs.

## 4. Calculating a Clearing Vector

For the case  $\Theta = \mathbf{0}_{n,n}$  Eisenberg and Noe (2001) develop an extremely elegant algorithm to calculate clearing vectors, the fictitious default algorithm. It has the nice feature that it reveals a sequence of defaults. In the first round of the algorithm it is assumed that the payments made equal the promised payments  $\bar{p}$ . Banks that are unable to meet their obligations are determined. These banks default even if all of their interbank claims are honored. In the next step the payments of these defaulting banks are adjusted such that they are in line with limited liability. If there are no additional defaults the iteration stops. If there are further defaults the procedure is continued. The important point is that the algorithm allows to distinguish between defaults that are directly related to adverse economic situations – exogenous income – and defaults that are caused by the defaults of other banks. The fictitious default algorithm works for  $e > \vec{0}$ . For  $e \in \mathbb{R}^n$  the algorithm might break down as is demonstrated in the next example.

**Example 2.** *To calculate a clearing vector for the case where  $\Theta = \mathbf{0}_{n,n}$  Eisenberg and Noe (2001) propose the following iterative procedure. Define the diagonal matrix  $\Lambda(p)$  by  $\Lambda_{ii}(p) = 1$  if  $e_i + \sum_{j=1}^n \Pi_{ji} p_j < \bar{p}_i$  and  $\Lambda_{ii}(p) = 0$  otherwise. Define the map  $p \rightarrow FF_{\hat{p}}(p)$  as follows:*

$$FF_{\hat{p}}(p) \equiv \Lambda(\hat{p})[\Pi'(\Lambda(\hat{p})p + (\mathbf{I} - \Lambda(\hat{p}))\bar{p}) + e] + (\mathbf{I} - \Lambda(\hat{p}))\bar{p}$$

*This map returns for all nodes not defaulting under  $\hat{p}$  the required payment  $\bar{p}$ . For all other nodes it returns the node's value assuming that non defaulting nodes pay  $\bar{p}$  and defaulting nodes pay  $p$ . Under suitable restrictions this map has a unique fixed point which is denoted by  $f(\hat{p})$ . Note that the equation for the fixed point*

$$f(\hat{p}) = \Lambda(\hat{p})[\Pi'(\Lambda(\hat{p})f(\hat{p}) + (\mathbf{I} - \Lambda(\hat{p}))\bar{p}) + e] + (\mathbf{I} - \Lambda(\hat{p}))\bar{p}$$

*can actually be written quite compactly as*

$$[\mathbf{I} - \Lambda(\hat{p})\Pi'\Lambda(\hat{p})](f(\hat{p}) - \bar{p}) = \Lambda(\hat{p})(e + \Pi'\bar{p} - \bar{p}). \quad (10)$$

Premultiplying by  $(\mathbf{I} - \Lambda(\hat{p}))$  yields

$$(\mathbf{I} - \Lambda(\hat{p}))(f(\hat{p}) - \bar{p}) = \vec{0}.$$

For banks that do not default  $\Lambda_{ii}(\hat{p}) = 0$  and  $f_i(\hat{p}) = \bar{p}_i$ . Premultiplying (10) by  $\Lambda(\hat{p})$  gives

$$\Lambda(\hat{p})(\mathbf{I} - \Pi')\Lambda(\hat{p})(f(\hat{p}) - \bar{p}) = \Lambda(\hat{p})(e + \Pi'\bar{p} - \bar{p}).$$

The  $ij$ th entry of  $\Lambda(\hat{p})(\mathbf{I} - \Pi')\Lambda(\hat{p})$  is zero unless  $\Lambda_{ii}(\hat{p}) = \Lambda_{jj}(\hat{p}) = 1$ . To calculate the fixed point, it suffices to consider the subsystem (submatrix) of defaulting nodes. The original system of equations can be chopped up into two independent systems. This is a major advantage if the number of nodes is large and default is a rare event.

Eisenberg and Noe (2001) show that under the assumption that  $e > \vec{0}$  (and  $\Theta = \mathbf{0}_{n,n}$ ) the sequence of payment vectors  $p^0 = \bar{p}$ ,  $p^i = f(p^{i-1})$  decreases to a clearing vector in at most  $n$  iterations. The assumption that  $e > \vec{0}$  is essential as is illustrated by the following example.

$$e = \begin{pmatrix} 1 \\ \frac{3}{4} \\ -\frac{9}{8} \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Setting  $p^0 = \bar{p}$  yields

$$e + \Pi'p^0 = \begin{pmatrix} \frac{9}{4} \\ \frac{3}{2} \\ -\frac{1}{8} \end{pmatrix}, \quad \Lambda(p^0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad p^1 = f(p^0) = \begin{pmatrix} 1 \\ -\frac{3}{20} \\ -\frac{6}{5} \end{pmatrix}.$$

Hence,  $p^1$  is not feasible and the algorithm breaks down. A possible remedy would be to use  $p^i = [f(p^{i-1}) \vee \vec{0}]$ . This procedure results in  $p^2 = p^1 = (1, 0, 0)'$ . It is easy to verify that  $(1, 0, 0)'$  is not a clearing vector. The unique clearing vector for the example is given by  $p^* = (1, \frac{3}{4}, 0)'$ .<sup>6</sup>

Even though the fictitious default algorithm might not work anymore it is still possible to define a simple but admittedly less elegant iterative procedure to calculate a clearing vector. We start with  $p^0 = \bar{p}$  and calculate the implied node values  $W^*(p^0)$ . This fix point can be determined in finitely many steps. If  $p^0$  is affordable, i.e.  $p^0 \leq [(W^*(p^0) + \bar{p}) \vee \vec{0}]$ , we are done. Otherwise the payments are reduced such that they are in line

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<sup>6</sup>The fictitious default algorithm works if each  $p^i$  is a supersolution. This can be guaranteed for  $e > \vec{0}$  and  $p^0 = \bar{p}$ . For  $e \not\geq \vec{0}$  this property may not hold.

with limited liability, i.e.  $p^1 = [(W^*(p^0) + \bar{p}) \vee \vec{0}] \wedge \bar{p}$ . The next theorem shows that this procedure converges to the largest clearing vector.

**Theorem 4.** *If  $\Theta$  is a holding matrix, the sequence  $p^{i+1} = [(W^*(p^i) + \bar{p}) \vee \vec{0}] \wedge \bar{p}$  started at  $p^0 = \bar{p}$  is well defined, decreasing, and converges to the largest clearing vector  $p^+$ .*

The proposed solution algorithm detects the same sequence of defaults as the fictitious default algorithm. Banks defaulting in the first round are those that default even if their claims are honored fully. Banks defaulting in later rounds are dragged into default by their interbank counter parties.

## 5. Comparative Statics

Without cross holdings the clearing vector is a concave function of  $e$  and  $\bar{p}$  as is shown by Eisenberg and Noe (2001). A simple example shows that the clearing vector is not concave as soon as cross holdings are included.

**Example 3.** *Assume the network is characterized by the following parameters:*

$$e = \begin{pmatrix} 0 \\ \lambda \\ -\frac{1}{10} \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix}$$

Table 1 shows the clearing vectors and equity values as functions of  $\lambda \in \mathbb{R}^n$ . Figure 1 illustrates that the clearing payments of banks 1 and 3 are not concave in  $\lambda$  and that the equity values of banks 1 and 2 are not convex in  $\lambda$ .

In Figure 1 the equity value of bank 2,  $V_2^*$ , increases disproportionately to exogenous income  $e_2$ , i.e.  $\frac{\partial V_2^*}{\partial e_2} > 1$ . Changing  $e_2$  from  $3/15$  to  $7/15$  increases the equity value of bank 2 from  $6/15$  to  $22/15$ . The value of bank 1's debt increases from 0.2 to 1 and the value of bank 3's debt increases from 0 to  $4/15$ . Hence, a subsidy of  $4/15$  increases both the unconsolidated equity value of the system and the value of the debt payments each by  $16/15$ . The value of the unconsolidated system increases by 8-times the original subsidy. This multiplier effect is very pronounced for  $e_2$  in the interval  $[3/15, 7/15]$  where banks 1 and 3 are in default. But even for comparatively large  $e$  where all three banks are solvent, changes in  $e_2$  cause a disproportionate change in the value of the unconsolidated system. Increasing  $e_2$  from 4 to 5 has no effect on debt payments as all banks are solvent but the equity value of the unconsolidated system increases by 1.75 from 7.65 to 9.4. Supposedly small shocks to  $e$  may have large effects on the debt and equity values.

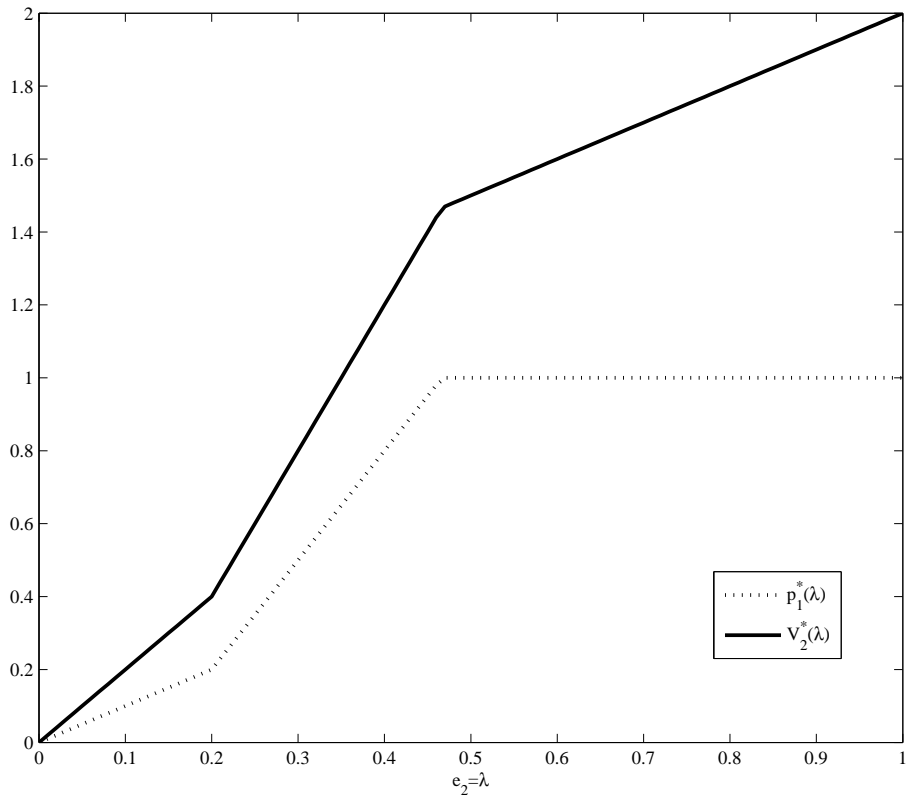


Figure 1: Debt payments of node 1 (dotted line) and the equity value of node 2 (solid line) as functions of  $e_2 = \lambda$  in Example 3. Debt payments are not concave and the equity value is not convex in  $\lambda$ .

$\lambda \leq 0$	$p^* = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$V^* = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
$\lambda \in [0, 0.2]$	$p^* = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}$	$V^* = \begin{pmatrix} 0 \\ 2\lambda \\ 0 \end{pmatrix}$
$\lambda \in [0.2, \frac{7}{15}]$	$p^* = \begin{pmatrix} 3\lambda - \frac{2}{5} \\ 0 \\ \lambda - \frac{1}{5} \end{pmatrix}$	$V^* = \begin{pmatrix} 0 \\ 4\lambda - \frac{2}{5} \\ 0 \end{pmatrix}$
$\lambda \in [\frac{7}{15}, \frac{17}{5}]$	$p^* = \begin{pmatrix} 1 \\ 0 \\ \frac{\lambda}{4} + \frac{3}{20} \end{pmatrix}$	$V^* = \begin{pmatrix} \frac{3}{4}\lambda - \frac{7}{20} \\ \lambda + 1 \\ 0 \end{pmatrix}$
$\lambda \geq \frac{17}{5}$	$p^* = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$	$V^* = \begin{pmatrix} \frac{1}{2}(\lambda + 1) \\ \lambda + 1 \\ \frac{1}{4}(\lambda + 1) - \frac{11}{10} \end{pmatrix}$

Table 1: Clearing vector and equity value as functions of banks 2's income  $\lambda$  in Example 3.

An alternative interpretation is that injecting money into the network increases the (unconsolidated) value of the claims by more than the supplied money. But how does this effect the wealth of outside investors? Let  $p^*(e)$  and  $V^*(p^*(e), e)$  be the clearing vector and the equity values corresponding to the financial network consisting of  $(e, \Pi, \bar{p}, \Theta)$ . Suppose some outside investor injects money into the network such that  $\tilde{e} \geq e$ . For  $e \geq \vec{0}$  we have

$$V^*(p^*(e), e) = e + (\Pi' - \mathbf{I})p^*(e) + \Theta'V^*(p^*(e), e)$$

or

$$(\mathbf{I} - \Theta')V^*(p^*(e), e) + (\mathbf{I} - \Pi')p^*(e) = e.$$

This implies that

$$\bar{\mathbf{I}}'(\mathbf{I} - \Theta')(V^*(p^*(\tilde{e}), \tilde{e}) - V^*(p^*(e), e)) + \bar{\mathbf{I}}'(\mathbf{I} - \Pi')(p^*(\tilde{e}) - p^*(e)) = \bar{\mathbf{I}}'(\tilde{e} - e).$$

By increasing  $e$  to  $\tilde{e}$  the value of an outside investor's portfolio changes by

$$\epsilon'(V^*(p^*(\tilde{e}), \tilde{e}) - V^*(p^*(e), e)) + \delta'(p^*(\tilde{e}) - p^*(e))$$

where  $\epsilon$  comprises the equity and  $\delta$  the debt holdings of the outside investor. Even

if the entire network is owned by a single outside investor, i.e.  $\delta' = \bar{\Gamma}'(\mathbf{I} - \Pi')$  and  $\epsilon' = \bar{\Gamma}'(\mathbf{I} - \Theta')$ , the amount gained will never exceed the amount injected, i.e.  $\bar{\Gamma}'(\bar{\epsilon} - e)$ . If there are no bankruptcy costs, it does never pay to bail out.

For the case where  $e \not\geq \bar{0}$  a part of the injected money is used to pay off more senior liabilities and as consequence the increase in consolidated values may be the less than  $\bar{\Gamma}'(\bar{\epsilon} - e)$ .

## 6. Seniority Structure

Eisenberg and Noe (2001) interpret  $e_i$  as exogenous operating cash flow. They restrict  $e_i$  to be nonnegative reasoning that any operating costs like wages can be captured by appending a "sink node" to the financial system. Such a sink node has no operating cash flow of its own, nor any obligations to other nodes. The implicit assumption is that the operating costs are of the same priority as the liabilities in the financial system. If these costs are of a higher priority, modeling them via a sink node is not correct.

**Example 4.** *Assume that the financial system consists of two banks. Bank 1 has an operating cash flow of 0.5. Bank 2 has revenues of 2 but has to pay wages of 4. In the interbank market bank 1 owes bank 2 one unit and vice versa. If wages have the same priority as the interbank liabilities, we append an additional node 3 to the system for the workers. So*

$$e = \begin{pmatrix} \frac{1}{2} \\ 2 \\ 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{5} & 0 & \frac{4}{5} \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}, \quad \Theta = \mathbf{0}_{3,3}.$$

*Clearing the system yields*

$$p^* = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \quad \text{and} \quad \Pi'p^* + e - p^* = \begin{pmatrix} \frac{1}{10} \\ 0 \\ \frac{24}{10} \end{pmatrix}.$$

*The shortfall of node 2 is proportionally shared between bank 1 and the workers. Each of them loses 40% of the promised payments. If we assume by contrast that wages are more senior than interbank claims, the sink node approach can not be used. Yet, the problem is still well defined and can be solved. The system*

$$e = \begin{pmatrix} \frac{1}{2} \\ -2 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



has the solution

$$p^* = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \quad \Pi' p^* + e - p^* = \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}$$

In this case node 1 is bankrupt. It loses all payments promised by node 2. Node 2 is not able to pay off its obligations to the workers either ( $e_2 + \Pi_{1,2} p_1^* < 0$ ). The workers lose 1.5 of the promised payments.

Finally, suppose there is a simple bilateral netting agreement between banks 1 and 2 which stipulates that crosswise nominal obligations are netted. In this case bank 1 has a value of 1/2 and the entire losses have to be borne by the workers. In this case interbank obligations would be of the highest priority.

The example demonstrates that the introduction of sink nodes is not as innocuous as it may seem. If there are different levels of priority, the amount available to pay off the most junior debt might be negative. Bank  $i$ 's income  $e_i$  in the previous sections can be interpreted as the amount available to pay off the most junior debt. This makes it necessary not to restrict  $e_i$  to be nonnegative.

To adapt the framework to a more elaborate seniority structure I introduce seniority classes. Different liabilities are in the same seniority class if – in case of default – repayment is rationed proportionally between them.<sup>7</sup> Each bank may have a different number of priority classes  $S_i$ . Let  $S^*$  be the maximum of these  $S_i$ . Assume that debt claims in class 1 are of the highest priority, i.e. have to be satisfied first, then the claims in class 2 sequentially up to class  $S^*$  are satisfied. Debt claims include interbank positions as well as obligations to parties outside the banking system such as depositors or bondholders. Denote by  $\bar{p}_{is} = \sum_{j=1}^N L_{ijs} + D_{is}$  the liabilities of bank  $i$  in class  $s$ . Define

$$\Pi_{ijs} = \begin{cases} \frac{L_{ijs}}{\bar{p}_{is}} & \text{if } \bar{p}_{is} > 0 \\ 0 & \text{otherwise} \end{cases}$$

and assume that if bank  $i$  has no debt in seniority class  $s$  then it has no debt in seniority class  $s + 1$  either ( $\bar{p}_{is} = 0$  implies  $\bar{p}_{is+1} = 0$ ). Let  $p_{\cdot s} = (p_{1s}, \dots, p_{ns})'$  and

$$\Pi_s = \begin{pmatrix} \Pi_{11s} & \cdots & \Pi_{1ns} \\ \vdots & \ddots & \vdots \\ \Pi_{n1s} & \cdots & \Pi_{nns} \end{pmatrix}.$$

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<sup>7</sup>Lando (2004, pp. 247) and Elsinger et al. (2006b) discuss the consequences of bilateral netting agreements in a network model. It is important to highlight that netting agreements can appropriately be taken into account only if different priority levels exist. As can be seen in Example 4 netting of nominal obligations is equivalent to assuming that the involved liabilities are of the highest priority.

In analogy to the case of just one seniority class equity values  $V^*(p)$  for a given  $p = (p_{11} \dots p_{1S^*}, p_{21} \dots p_{2S^*}, \dots, p_{n1} \dots p_{nS^*})$  are defined as a fixed point of  $\Psi^1(\cdot; p, e, \Pi, \Theta) : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$

$$V^*(p) = [e + \sum_{s=1}^{S^*} (\Pi_s)' p_{\cdot s} - \sum_{s=1}^{S^*} p_{\cdot s} + \Theta' V^*(p)] \vee \vec{0}.$$

Lemma 4 guarantees that  $V^*(p)$  exists and is unique given that  $\Theta$  is a holding matrix. A clearing payment vector has to satisfy limited liability and the seniority structure of the liabilities including absolute priority of debt.

**Definition 2.**  $p^* \geq \vec{0}$  is a clearing vector if  $\forall i \in \{1, \dots, n\}$  and  $\forall T \in \{1, \dots, S^*\}$

$$p_{iT}^* = \min \left( \max \left( e_i + \sum_{j=1}^N \sum_{s=1}^{S^*} \Pi_{jis} p_{js}^* - \sum_{s=1}^{T-1} p_{is}^* + \sum_{j=1}^N \Theta_{ji} V_j^*(p^*), 0 \right), \bar{p}_{iT} \right).$$

The definition insures that debt is paid off according to seniority. A clearing vector has the property that if debt in seniority class  $T$  is not fully honored ( $p_{iT}^* < \bar{p}_{iT}$ ), debt in seniority class  $T + 1$  is not served at all ( $p_{i(T+1)}^* = 0$ ). On the other hand repaying at least a fraction of the debt in seniority class  $T + 1$ , i.e.  $p_{i(T+1)}^* > 0$ , implies that debt in seniority class  $T$  is repaid in full. As a consequence of a detailed priority structure  $e_i$  might as well be assumed to be nonnegative.

The introduction of a detailed seniority structure does not change the main results which are summarized in the sequel. In analogy to the case of only one seniority class a clearing vector can be defined as a fixed point of the map

$\Phi^1(p) = (\Phi_{11}^1 \dots \Phi_{1S^*}^1, \Phi_{21}^1 \dots \Phi_{2S^*}^1, \dots, \Phi_{n1}^1 \dots \Phi_{nS^*}^1)' : [\vec{0}, \bar{p}] \rightarrow [\vec{0}, \bar{p}]$  defined by

$$\Phi_{iT}^1(p) = \left\{ \left[ e_i + \sum_{j=1}^N \sum_{s=1}^{S^*} \Pi_{jis} p_{js} - \sum_{s=1}^{T-1} p_{is} + \sum_{j=1}^N \Theta_{ji} V_j^*(p) \right] \vee 0 \right\} \wedge \bar{p}_{iT}. \quad (11)$$

Let

$$W^*(p) = e + \sum_{s=1}^{S^*} (\Pi_s)' p_{\cdot s} - \sum_{s=1}^{S^*} \bar{p}_{\cdot s} + \Theta'(W^*(p) \vee \vec{0}) \quad (12)$$

and

$$\Phi_{iT}^2(p) = \left\{ \left[ e_i + \sum_{j=1}^N \sum_{s=1}^{S^*} \Pi_{jis} p_{js} - \sum_{s=1}^{T-1} \bar{p}_{is} + \sum_{j=1}^N \Theta_{ji} (W_j^*(p) \vee 0) \right] \vee 0 \right\} \wedge \bar{p}_{iT}. \quad (13)$$

**Theorem 1'.** If  $\hat{p} \in [\vec{0}, \bar{p}]$  is a (super)solution of  $\Phi^1(p)$ , i.e.  $\hat{p} \geq \Phi^1(\hat{p}; \Pi, \bar{p}, e, \Theta)$ , then

$\hat{p}$  is a (super)solution of  $\Phi^2(p)$  with

$$W^*(p) = e + \sum_{s=1}^{S^*} (\Pi_s)' p_{\cdot s} - \sum_{s=1}^{S^*} \bar{p}_{\cdot s} + \Theta' V^*(p)$$

and vice versa with  $V^*(\hat{p}) = (W^*(\hat{p}) \vee \vec{0})$ .

**Theorem 2'.** There exists a greatest ( $p^+$ ) and a least ( $p^-$ ) clearing vector.

**Theorem 3'.** Suppose the network allows for multiple clearing vectors  $p^+ \neq p^-$  with corresponding equity values  $V^+$  and  $V^-$ . Let  $\mathcal{I}^0$  be the subset of banks with non-unique equity values and  $\mathcal{I}^s$  be the subset of banks with non-unique clearing payments in seniority class  $s$ . Let  $\mathcal{I} = \bigcup_{s=0}^{S^*} \mathcal{I}^s$ . It has to hold that

1. all banks in  $\mathcal{I}^0$  are entirely owned by banks in  $\mathcal{I}$ , i.e.  $\sum_{j \in \mathcal{I}} \Theta_{ij} = 1$  for all  $i \in \mathcal{I}^0$ ,
2. the only creditors in seniority class  $s$  of banks in  $\mathcal{I}^s$  are banks in  $\mathcal{I}$ , i.e.  $\sum_{j \in \mathcal{I}} \Pi_{ijs} = 1$  for all  $i \in \mathcal{I}^s$ , and
3. the sum of the obligations of banks in  $\mathcal{I}$  to banks not in  $\mathcal{I}$  has to equal the aggregate exogenous income of banks in  $\mathcal{I}$  plus the value of all claims of banks in  $\mathcal{I}$  against banks not in  $\mathcal{I}$ , i.e.

$$\sum_{s=1}^{S^*} \sum_{i \in \mathcal{I}} \left( 1 - \sum_{j \in \mathcal{I}} \Pi_{ijs} \right) p_{is}^+ = \sum_{i \in \mathcal{I}} e_i + \sum_{s=1}^{S^*} \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Pi_{ijs} \right) p_{is}^* + \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Theta_{ij} \right) V_i^*$$

where  $p_{is}^* = p_{is}^+ = p_{is}^-$  and  $V_i^* = V_i^+ = V_i^-$  for  $i \notin \mathcal{I}$ . If there is only one seniority class this boils down to

$$\sum_{i \in \mathcal{I}} \left( 1 - \sum_{j \in \mathcal{I}} \Pi_{ij1} \right) \bar{p}_{i1} = \sum_{i \in \mathcal{I}} e_i + \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Pi_{ij1} \right) p_{i1}^* + \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Theta_{ij} \right) V_i^*.$$

To calculate a clearing vector we start with  $p^0 = \bar{p}$  and calculate  $W^*(p^0)$ . If

$$p_{iT}^0 \leq \left\{ \left[ W_i^*(p^0) + \sum_{s=T}^{S^*} \bar{p}_{is} \right] \vee 0 \right\} \wedge \bar{p}_{iT} \quad \forall i \in \{1, \dots, n\} \quad \text{and} \quad \forall T \in \{1, \dots, S^*\}$$

we are done. Otherwise we set

$$p_{iT}^1 = \left\{ \left[ W_i^*(p^0) + \sum_{s=T}^{S^*} \bar{p}_{is} \right] \vee 0 \right\} \wedge \bar{p}_{iT}$$

and iterate the procedure.  $W^*(p)$  is increasing in  $p$  implying that  $p^1 \leq p^0$  for all  $k$  and  $\lim_{k \rightarrow \infty} p^k = p^+$  where  $p^+$  denotes the largest clearing vector.

**Theorem 4.** If  $\Theta$  is a holding matrix, the sequence

$$p_{iT}^k = \left\{ \left[ W_i^*(p^{k-1}) + \sum_{s=T}^{S^*} \bar{p}_{is} \right] \vee 0 \right\} \wedge \bar{p}_{iT}$$

started at  $p^0 = \bar{p}$  is well defined, decreasing, and converges to the largest clearing vector  $p^+$ .

**Example 5.** Using different priority classes we may rewrite Example 3 to show that even if  $e > 0$ ,  $p^*$  is not necessarily concave in  $e$ . To do this interpret  $e$  in Example 3 as the net position after subtracting high priority debt from income. Assume that the counterparties of the highest priority debt are not part of the network. These liabilities are given by  $D_{\cdot 1} = (D_{11}, D_{21}, D_{31})' = (1, 1, \frac{11}{10})'$ . There are no liabilities of the same priority within the network and hence  $L_1 = \mathbf{0}_{3,3}$ . Let  $e = (1, 1+\lambda, 1)'$ . So the net position after clearing highest seniority debt is equal to  $e$  in Example 3. The other parameters are given by  $D_{\cdot 2} = \vec{0}$ ,

$$L_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ and } \Theta = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}.$$

Hence,  $\bar{p} = (1, 1, \frac{11}{10}, 1, 0, 1)$ . It is easy to verify that the clearing payments of node 1 in seniority class 2 equal the clearing payment of node 1 in Example 3. Hence,  $p^*$  is not concave in  $e$ .

## 7. Conclusions

In this paper I analyze networks of financial institutions that are linked with each other via debt and equity claims. Limited liability of equity and a detailed seniority structure of debt are taken into account explicitly. The values of these claims are finite but not necessarily unique. Yet, whenever an outside investor holds a claim against a bank the value of this claim is unique. By adjusting the fictitious default algorithm developed in Eisenberg and Noe (2001) debt and equity values can be determined.

As long as bankruptcy costs are zero it never pays to bail out insolvent banks. Introducing bankruptcy costs does not change the main result that a solution to the clearing problem exists. Yet, bailing out insolvent banks or forgiving debt may become profitable.

The model is static. But by the inclusion of a detailed seniority structure it allows to take the timing of the payments into account and has thereby a dynamic flavor. The contagion effects of a negative shock to the economy (low realization of  $e$ ) can be analyzed more precisely than in the case were all liabilities are modeled as *pari passu*.

The model presented is part of a simulation software at the Oesterreichische Nationalbank to assess the stability of the Austrian banking sector.<sup>8</sup> The simulations rest on the assumption that exogenous income  $e$  is a multidimensional random variable. For each draw of  $e$  the system is cleared. If  $e$  is drawn from the objective distribution, default probabilities and contagion effects can be assessed. If  $e$  is drawn from the risk neutral measure, the value of debt and equity can be determined by averaging across the simulations.

Modeling correlated defaults is not only an issue in the banking literature but also in the literature on the valuation of complex portfolio credit derivatives such as collateralized debt obligations (CDOs). The value of the different CDO tranches depends crucially on the joint distribution of default of the underlying collateral securities. The linkages between these securities (obligors) are not modeled explicitly but via the assumption that the default intensities of these securities are correlated (Duffie and Gârleanu (2001), Longstaff and Rajan (2008), and Errais et al. (2009)).

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<sup>8</sup>A detailed description of the software is given in a technical document (Boss et al. (2006)) which is available upon request.

## References

- Allen, Franklin, Douglas Gale. 2000. Financial contagion. *Journal of Political Economy* **108**(1) 1–34.
- Boss, Michael, Thomas Breuer, Helmut Elsinger, Gerald Krenn, Alfred Lehar, Claus Pühr, Martin Summer. 2006. Systemic risk monitor: Risk assessment and stress testing for the austrian banking system. model documentation. Tech. rep., Oesterreichische Nationalbank.
- Boyd, John H., Sungkyu Kwak, Bruce Smith. 2005. The real output losses associated with modern banking crises. *Journal of Money, Credit and Banking* **37**(6) 977–999.
- Duffie, Darrell, Nicolae B. Gârleanu. 2001. Risk and valuation of collateralized debt obligations. *Financial Analysts Journal* **57** 41–59.
- Eisenberg, Larry, Thomas Noe. 2001. Systemic risk in financial systems. *Management Science* **47**(2) 236–249.
- Elsinger, Helmut, Alfred Lehar, Martin Summer. 2006a. Risk assessment for banking systems. *Management Science* **52** 1301–1314.
- Elsinger, Helmut, Alfred Lehar, Martin Summer. 2006b. Using market information for banking system risk assessment. *International Journal of Central Banking* **2**(1) 137–165.
- Errais, Eymen, Kay Giesecke, Lisa R. Goldberg. 2009. Affine point processes and portfolio credit risk. Mimeo.
- Furfine, Craig. 2003. Interbank exposures: Quantifying the risk of contagion. *Journal of Money, Credit, and Banking* **35**(1) 111–128.
- Giesecke, Kay, Stefan Weber. 2004. Cyclical correlations, credit contagion and portfolio losses. *Journal of Banking and Finance* **28**(12) 3009–3036.
- Lando, David. 2004. *Credit Risk Modeling*. Princeton University Press.
- Longstaff, Francis A., Arvind Rajan. 2008. An empirical analysis of the pricing of collateralized debt obligations. *Journal of Finance* **63** 509–563.
- Mistrulli, Paolo E. 2006. Assessing financial contagion in the interbank market: A comparison between estimated and observed bilateral exposures. Mimeo.
- Rochet, Jean-Charles, Jean Tirole. 1997. Interbank lending and systemic risk. *Journal of Money, Credit, and Banking* **28**(4) 733–762.
- Shin, Hyun Song. 2008. Risk and liquidity in a system context. *Journal of Financial Intermediation* **17**(3) 315 – 329.
- Upper, Christian. 2007. Using counterfactual simulations to assess the danger of contagion in interbank markets. BIS, Working Paper 234.
- Zeidler, Eberhard. 1986. *Nonlinear Functional Analysis and its Applications I*. Springer Verlag.

## A. Proofs

**Definition 3.** Let  $y$  and  $x$  be  $n \times 1$  vectors. Then  $\Lambda := \text{diag}(y \geq x)$  is an  $n \times n$  diagonal matrix where  $\Lambda_{ii} = 1$  if  $y_i \geq x_i$  and  $\Lambda_{ii} = 0$  otherwise.  $\text{diag}(y > x)$ ,  $\text{diag}(y \leq x)$ ,  $\text{diag}(y < x)$ ,  $\text{diag}(y \neq x)$ , and  $\text{diag}(y = x)$  are defined analogously.

**Lemma 1.** Let  $\Theta \in [0, 1]^{n \times n}$  be a matrix of interbank share holdings and let  $\mathbf{I}$  be the  $n \times n$  identity matrix.  $(\mathbf{I} - \Theta')$  is invertible if and only if Assumption 1 is satisfied, i.e.  $\Theta$  is a holding matrix.

*Proof.* It suffices to show that  $(\mathbf{I} - \Theta')$  is invertible. Assume that there is a subset  $\mathcal{I} \subset \{1, \dots, n\}$  such that  $\sum_{j \in \mathcal{I}} \Theta_{ij} = 1$  for all  $i \in \mathcal{I}$ . Let  $x$  be an  $n \times 1$  vector with components  $x_i = 1$  if  $i \in \mathcal{I}$  and  $x_i = 0$  otherwise. Clearly,  $(\mathbf{I} - \Theta)x = \vec{0}$  where  $\vec{0}$  denotes the  $n \times 1$  dimensional zero vector. Thus  $(\mathbf{I} - \Theta)$  is not invertible.

Now assume that  $(\mathbf{I} - \Theta)$  is not invertible. Then there exists a vector  $x \neq \vec{0}$  such that  $(\mathbf{I} - \Theta)x = \vec{0}$ . Writing down this system equation by equation we have a linear system given by

$$x_i = \sum_{j=1}^n \Theta_{ij} x_j \quad \text{for } i = 1, \dots, n.$$

Taking absolute values on both sides and applying the triangle inequality yields

$$|x_i| = \left| \sum_{j=1}^n \Theta_{ij} x_j \right| \leq \sum_{j=1}^n \Theta_{ij} |x_j| \quad \text{for } i = 1, \dots, n.$$

Now construct an index set  $\mathcal{I} \subset \{1, \dots, n\}$  as follows. The index  $i$  is in  $\mathcal{I}$  if and only if  $|x_i| \geq |x_j|$  for  $j = 1, \dots, n$ . Since the triangle inequality holds for all  $i$  it holds in particular for all  $i \in \mathcal{I}$ . Thus we have

$$|x_i| \leq \sum_{j=1}^n \Theta_{ij} |x_j| \leq |x_i| \left( \sum_{j \in \mathcal{I}} \Theta_{ij} + \sum_{j \notin \mathcal{I}} \Theta_{ij} \right) \leq |x_i| \quad \text{for all } i \in \mathcal{I}$$

with equality only if  $\sum_{j \in \mathcal{I}} \Theta_{ij} = 1$ . Hence, if  $(\mathbf{I} - \Theta)$  is not invertible it has to hold that  $\sum_{j \in \mathcal{I}} \Theta_{ij} = 1$  for all  $i \in \mathcal{I}$ . This violates Assumption 1.  $\square$

**Lemma 2.** Let  $\Theta$  be an  $n \times n$  holding matrix, let  $u$  be a  $n \times 1$  vector, and let  $\Lambda = \text{diag}(u > \vec{0})$ . If  $\Lambda \neq \mathbf{0}_{n,n}$  where  $\mathbf{0}_{n,n}$  is an  $n \times n$  matrix of zeros, it holds that  $\vec{1}' \Lambda (\mathbf{I} - \Theta') \Lambda u > 0$ . Analogously, if  $\Lambda = \text{diag}(u < \vec{0})$  then  $\vec{1}' \Lambda (\mathbf{I} - \Theta') \Lambda u > 0$ .

*Proof.* Let  $\Lambda = \text{diag}(u > \vec{0})$ .  $\Lambda$  is idempotent. Hence,

$$\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda u = \vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda\Lambda u$$

$\Lambda u \geq \vec{0}$  by construction.  $\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda \geq \vec{0}'$  as no row sum of  $\Theta$  exceeds one. This implies that  $\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda u \geq 0$ . Now, suppose  $\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda u = 0$  and define the index set  $\mathcal{I} := \{i | u_i > 0\}$ .  $\Lambda \neq \mathbf{0}_{n,n}$  implies that  $\mathcal{I}$  is not empty. It has to hold that

$$0 = \sum_{i \in \mathcal{I}} u_i - \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \Theta_{ji} u_j = \sum_{i \in \mathcal{I}} u_i - \sum_{j \in \mathcal{I}} \left( \sum_{i \in \mathcal{I}} \Theta_{ji} \right) u_j$$

This implies that  $\sum_{i \in \mathcal{I}} \Theta_{ji} = 1$  for all  $j \in \mathcal{I}$ . But this violates Assumption 1. The second part of the lemma can be proved analogously.  $\square$

**Lemma 3.** *Let  $\Theta$  be a  $n \times n$  holding matrix and let  $y \geq \vec{0}$  be a  $n \times 1$  vector. Then there exists a unique  $x \geq y$  such that  $x = y + \Theta'x$ .*

*Proof.* Lemma 1 implies that  $x$  is unique. Now, suppose  $x \not\geq y$  and let  $\Lambda = \text{diag}(x < y)$ .

$$\Lambda x = \Lambda y + \Lambda\Theta'\Lambda x + \Lambda\Theta'(\mathbf{I} - \Lambda)x.$$

By construction  $(\mathbf{I} - \Lambda)x \geq (\mathbf{I} - \Lambda)y$ . So the last equation may be rewritten as

$$\Lambda(\mathbf{I} - \Theta')\Lambda(x - y) \geq \Lambda\Theta'y.$$

Premultiplying by  $\vec{1}'$  yields

$$\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda(x - y) \geq \vec{1}'\Lambda\Theta'y \geq 0.$$

If  $\Lambda \neq \mathbf{0}_{n,n}$  the left hand side is smaller than 0 by Lemma 2. Therefore,  $x \geq y$ .  $\square$

**Lemma 4.** *Let  $u \in \mathbb{R}^n$  and  $\Theta$  be a holding matrix. Then the map  $F(\cdot; u) : \mathbb{R}^n \rightarrow \mathbb{R}_+^n$*

$$F(V; u) = [u + \Theta'V] \vee \vec{0} \tag{14}$$

*has a unique fixed point,  $V^* \geq \vec{0}$ .*

*Proof.* Define  $\tilde{V}$  by  $\tilde{V} = [u \vee \vec{0}] + \Theta'\tilde{V}$ . Given that  $\Theta$  is a holding matrix, Lemma 1 implies that  $\tilde{V}$  is well defined and unique. Lemma 3 implies  $\tilde{V} \geq [u \vee \vec{0}] \geq \vec{0}$ . Moreover,

$$F(\tilde{V}; u) = [u + \Theta'\tilde{V}] \vee \vec{0} = [u - (u \vee \vec{0}) + \tilde{V}] \vee \vec{0} \leq \tilde{V}.$$



As  $F(V; u)$  is increasing on the complete lattice  $[\vec{0}, \vec{V}]$  the Tarski fixed point theorem (Theorem 11.E in Zeidler (1986)) implies that there exists a greatest and a least fixed point,  $V^+$  and  $V^-$ , in the interval  $[\vec{0}, \vec{V}]$ . Suppose  $V^*$ , not necessarily in  $[\vec{0}, \vec{V}]$ , is an arbitrary fixed point of  $F(\cdot; u)$ . Let  $\Lambda = \text{diag}(V^* > V^-)$ . Note that  $\Lambda V^* = \Lambda(u + \Theta'V^*)$  and  $\Lambda V^- \geq \Lambda(u + \Theta'V^-)$ . This implies that

$$\Lambda(V^* - V^-) \leq \Lambda\Theta'(\Lambda(V^* - V^-) + (\mathbf{I} - \Lambda)(V^* - V^-)).$$

Rearranging and premultiplying by  $\vec{1}'$  yields

$$\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda(V^* - V^-) \leq \vec{1}'\Lambda\Theta'(\mathbf{I} - \Lambda)(V^* - V^-).$$

The right hand side of the above inequality is less than or equal to 0. Lemma 2 implies that the left hand side is larger than 0 as long as  $\Lambda \neq \mathbf{0}_{n,n}$ . So it has to hold that  $V^* \leq V^-$ . Evidently,  $V^* \geq \vec{0}$ . But as  $V^-$  is the smallest fixed point in  $[\vec{0}, \vec{V}]$  it follows that  $V^* = V^-$  and the fixed point is unique.  $\square$

**Lemma 5.** *Let  $\Theta$  be a holding matrix. Then*

$$W = u + \Theta'(W \vee \vec{0}) \tag{15}$$

has a unique solution  $W^*$  for any  $n \times 1$  vector  $u$ . If  $u^1$  and  $u^2$  are two  $n \times 1$  vectors such that  $u^2 \geq u^1$  then  $W^*(u^2) - W^*(u^1) \geq u^2 - u^1$  and  $(\mathbf{I} - \Theta)(W^*(u^2) - W^*(u^1)) \leq u^2 - u^1$  where  $W^*(u^1)$  and  $W^*(u^2)$  are the respective fixed points.

*Proof.* Lemma 4 establishes that  $V^* = [u + \Theta'V^*] \vee \vec{0}$  is unique. Let  $X = u + \Theta'V^*$ . It is easy to see that  $X$  solves (15). On the other hand if  $W^*$  solves (15) then  $X = [W^* \vee \vec{0}]$  is a fixed point of  $F(\cdot; u)$ . To prove uniqueness assume there exist two solutions,  $W^1$  and  $W^2$ . As  $F(\cdot; u)$  has a unique fixed point,  $[W^1 \vee \vec{0}] = [W^2 \vee \vec{0}]$ . But this in turn implies that  $W^1 = W^2$ . Hence, Equation (15) has a unique solution.

To prove the second claim assume that  $u^1$  and  $u^2$  are two vectors in  $\mathbb{R}^n$  such that  $u^2 \geq u^1$ . Let  $W^*(u^1) = u^1 + \Theta'(W^*(u^1) \vee \vec{0})$  and  $W^*(u^2) = u^2 + \Theta'(W^*(u^2) \vee \vec{0})$  be the respective fixed points. Let  $x = u^2 - u^1 \geq \vec{0}$ . It holds that

$$W^*(u^2) - W^*(u^1) = x + \Theta'([W^*(u^2) \vee \vec{0}] - [W^*(u^1) \vee \vec{0}])$$

Let  $\Lambda = \text{diag}(W^*(u^1) > W^*(u^2))$ . Note that  $\Lambda([W^*(u^2) \vee \vec{0}] - [W^*(u^1) \vee \vec{0}]) \geq \Lambda(W^*(u^2) -$

$W^*(u^1)$ ) and  $(I - \Lambda)([W^*(u^2) \vee \vec{0}] - [W^*(u^1) \vee \vec{0}]) \geq \vec{0}$ . Hence,

$$\Lambda(W^*(u^2) - W^*(u^1)) \geq \Lambda x + \Lambda \Theta \Lambda(W^*(u^2) - W^*(u^1))$$

Rearranging and premultiplying by  $\vec{1}'$  yields

$$\vec{1}' \Lambda (\mathbf{I} - \Theta') \Lambda (W^*(u^2) - W^*(u^1)) \geq \vec{1}' \Lambda x$$

The right hand side is larger or equal to 0. The left hand side is smaller than 0 by Lemma 2 unless  $\Lambda = \mathbf{0}_{n,n}$ . Hence,  $W^*(u^2) \geq W^*(u^1)$ . This implies

$$W^*(u^2) - W^*(u^1) \geq ([W^*(u^2) \vee \vec{0}] - [W^*(u^1) \vee \vec{0}]) \geq 0$$

and therefore  $W^*(u^2) - W^*(u^1) \geq u^2 - u^1$  and  $(\mathbf{I} - \Theta')(W^*(u^2) - W^*(u^1)) \leq u^2 - u^1$ .  $\square$

The remaining theorems are proved for a detailed seniority structure. This needs some additional notation. In analogy to the case of only one seniority class a clearing vector can be defined as a fixed point of the map

$\Phi^1(p) = (\Phi_{11}^1 \dots \Phi_{1S^*}^1, \Phi_{21}^1 \dots \Phi_{2S^*}^1, \dots, \Phi_{n1}^1 \dots \Phi_{nS^*}^1)' : [\vec{0}, \vec{p}] \rightarrow [\vec{0}, \vec{p}]$  defined by

$$\Phi_{iT}^1(p) = \left\{ \left[ e_i + \sum_{j=1}^N \sum_{s=1}^{S^*} \Pi_{jis} p_{js} - \sum_{s=1}^{T-1} p_{is} + \sum_{j=1}^N \Theta_{ji} V_j^*(p) \right] \vee 0 \right\} \wedge \bar{p}_{iT}. \quad (16)$$

Let

$$W^*(p) = e + \sum_{s=1}^{S^*} (\Pi_s)' p_{\cdot s} - \sum_{s=1}^{S^*} \bar{p}_{\cdot s} + \Theta'(W^*(p) \vee \vec{0}) \quad (17)$$

and

$$\Phi_{iT}^2(p) = \left\{ \left[ e_i + \sum_{j=1}^N \sum_{s=1}^{S^*} \Pi_{jis} p_{js} - \sum_{s=1}^{T-1} \bar{p}_{is} + \sum_{j=1}^N \Theta_{ji} (W_j^*(p) \vee 0) \right] \vee 0 \right\} \wedge \bar{p}_{iT}. \quad (18)$$

**Theorem 1.** If  $\hat{p} \in [\vec{0}, \vec{p}]$  is a (super)solution of  $\Phi^1(p)$ , i.e.  $\hat{p} \geq \Phi^1(\hat{p}; \Pi, \bar{p}, e, \Theta)$ , then  $\hat{p}$  is a (super)solution of  $\Phi^2(p)$  with

$$W^*(p) = e + \sum_{s=1}^{S^*} (\Pi_s)' p_{\cdot s} - \sum_{s=1}^{S^*} \bar{p}_{\cdot s} + \Theta' V^*(p)$$

and vice versa with  $V^*(\hat{p}) = (W^*(\hat{p}) \vee \vec{0})$ .

*Proof.* I prove the assertion for the case of supersolutions. The proof for solutions is analogous. Suppose that  $\hat{p}$  is a supersolution of  $\Phi^1$ , i.e.  $\hat{p} \geq \Phi^1(\hat{p})$ . Let  $X = e + \sum_{s=1}^{S^*} (\Pi_s)' \hat{p}_s - \sum_{s=1}^{S^*} \bar{p}_s + \Theta' V^*(\hat{p})$ . Evidently,  $V^*(\hat{p}) \geq X$ . Whenever,  $V_i^*(\hat{p}) > 0$  it has to hold that

$$0 < e_i + \sum_{j=1}^N \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{S^*} \hat{p}_{is} + \sum_{j=1}^N \Theta_{ji} V_j^*(p).$$

This implies that

$$\hat{p}_{iT} < e_i + \sum_{j=1}^N \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{T-1} \hat{p}_{is} + \sum_{j=1}^N \Theta_{ji} V_j^*(p) \quad \forall T \in \{1, \dots, S^*\}.$$

Given that  $\hat{p}$  is a supersolution of  $\Phi^1$  we get  $\hat{p}_{iT} = \bar{p}_{iT}$  for all  $T \in \{1, \dots, S^*\}$ . Therefore,  $V^*(\hat{p}) = (X \vee \vec{0})$  and  $X$  solves (17). As  $\Phi^1(\hat{p}) \geq \Phi^2(\hat{p})$  the vector  $\hat{p}$  is a supersolution of  $\Phi^2$ , too. Now, assume that  $\hat{p}$  is a supersolution of  $\Phi^2$ . Let  $X = (W^*(\hat{p}) \vee \vec{0})$ . If  $W_i^*(\hat{p}) \geq 0$  then

$$\bar{p}_{iT} \leq e_i + \sum_{j=1}^N \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{T-1} \bar{p}_{is} + \sum_{j=1}^N \Theta_{ji} (W_j^*(p) \vee 0) \quad \forall T \in \{1, \dots, S^*\}$$

implying that  $\hat{p}_{iT} = \bar{p}_{iT}$  for all  $T \in \{1, \dots, S^*\}$ . Hence, for all  $i$  with  $W_i^*(\hat{p}) \geq 0$  we get

$$X_i = e_i + \sum_{j=1}^N \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{S^*} \hat{p}_{is} + \sum_{j=1}^N \Theta_{ji} X_j.$$

Suppose  $W_i^*(\hat{p}) < 0$  and let  $H_i$  be the highest index such that  $\hat{p}_{iH_i} = \bar{p}_{iH_i}$ . If  $H_i = S^*$  then

$$0 > e_i + \sum_{j=1}^N \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{S^*} \hat{p}_{is} + \sum_{j=1}^N \Theta_{ji} X_j.$$

For  $H_i < S^*$  it has to hold that

$$\bar{p}_{iH_i+1} > \hat{p}_{iH_i+1} \geq e_i + \sum_{j=1}^N \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{H_i} \hat{p}_{is} + \sum_{j=1}^N \Theta_{ji} X_j \quad (19)$$

as  $\hat{p}$  is a supersolution. Hence,

$$0 \geq e_i + \sum_{j=1}^N \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{S^*} \hat{p}_{is} + \sum_{j=1}^N \Theta_{ji} X_j.$$

So we may write

$$X = \left( e + \sum_{s=1}^{S^*} \Pi'_s \hat{p}_{\cdot s} - \sum_{s=1}^{S^*} \hat{p}_{\cdot s} + \Theta' X \right) \vee \vec{0}.$$

To prove that  $\hat{p}$  is a supersolution of  $\Phi^1$  note that for all  $s \leq H_i + 1$ ,  $\Phi_{is}^1 = \Phi_{is}^2$  and for all  $s > H_i + 1$ , it has to hold that  $0 \geq \Phi_{is}^1 \geq \Phi_{is}^2$  as can be seen by (19). The vector  $\hat{p}$  is indeed a supersolution of  $\Phi^1$ .  $\square$

**Theorem 2.** There exists a greatest ( $p^+$ ) and a least ( $p^-$ ) clearing vector.

*Proof.* The last theorem establishes that any fixed point of  $\Phi^2(p)$  is a fixed point of  $\Phi^1(p)$  and vice versa. To prove that a clearing vector exists it suffices to show that  $\Phi^2(p)$  has a fixed point. By construction  $\Phi^2(\vec{0}) \geq \vec{0}$  and  $\Phi^2(\bar{p}) \leq \bar{p}$ . The Tarski fixed point theorem guarantees that there exists a least and a greatest fixed point for  $\Phi^2(p)$  if  $\Phi^2(p)$  is a monotone increasing function on the complete lattice  $[0, \bar{p}]$ . Lemma 5 shows that  $W^*(p)$  and thereby  $\Phi^2(p)$  are increasing in  $p$ .  $\square$

**Theorem 3.** Suppose the network allows for multiple clearing vectors  $p^+ \neq p^-$  with corresponding equity values  $V^+$  and  $V^-$ . Let  $\mathcal{I}^0$  be the subset of banks with non-unique equity values and  $\mathcal{I}^s$  be the subset of banks with non-unique clearing payments in seniority class  $s$ . Let  $\mathcal{I} = \bigcup_{s=0}^{S^*} \mathcal{I}^s$ . It has to hold that

1. all banks in  $\mathcal{I}^0$  are entirely owned by banks in  $\mathcal{I}$ , i.e.  $\sum_{j \in \mathcal{I}} \Theta_{ij} = 1$  for all  $i \in \mathcal{I}^0$ ,
2. the only creditors in seniority class  $s$  of banks in  $\mathcal{I}^s$  are banks in  $\mathcal{I}$ , i.e.  $\sum_{j \in \mathcal{I}} \Pi_{ijs} = 1$  for all  $i \in \mathcal{I}^s$ , and
3. the sum of the obligations of banks in  $\mathcal{I}$  to banks not in  $\mathcal{I}$  has to equal the aggregate exogenous income of banks in  $\mathcal{I}$  plus the value of all claims of banks in  $\mathcal{I}$  against banks not in  $\mathcal{I}$ , i.e.

$$\sum_{s=1}^{S^*} \sum_{i \in \mathcal{I}} \left( 1 - \sum_{j \in \mathcal{I}} \Pi_{ijs} \right) p_{is}^+ = \sum_{i \in \mathcal{I}} e_i + \sum_{s=1}^{S^*} \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Pi_{ijs} \right) p_{is}^* + \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Theta_{ij} \right) V_i^*$$

where  $p_{is}^* = p_{is}^+ = p_{is}^-$  and  $V_i^* = V_i^+ = V_i^-$  for  $i \notin \mathcal{I}$ . If there is only one seniority class this boils down to

$$\sum_{i \in \mathcal{I}} \left( 1 - \sum_{j \in \mathcal{I}} \Pi_{ij1} \right) \bar{p}_{i1} = \sum_{i \in \mathcal{I}} e_i + \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Pi_{ij1} \right) p_{i1}^* + \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Theta_{ij} \right) V_i^*.$$

*Proof.* Let  $p^-$  be the least and  $p^+$  be the largest clearing vector with  $p^+ \neq p^-$ . Denote the corresponding equity values by  $V^-$  and  $V^+$ . Let  $\Lambda^0 = \text{diag}(V^+ > V^-)$  and  $\Lambda^s = \text{diag}(p_{.s}^+ > p_{.s}^-)$  for each  $s \in \{1, \dots, S^*\}$ . Let  $\Lambda$  characterize all banks that either belong to  $\Lambda^0$  or some  $\Lambda^s$ , i.e.  $\Lambda = \mathbf{I} - \prod_{s=0}^{S^*} (\mathbf{I} - \Lambda^s)$ . Subtracting  $V^-$  from  $V^+$  and multiplying by  $\vec{1}'\Lambda$  yields

$$\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda^0(V^+ - V^-) \leq \sum_{s=1}^{S^*} \vec{1}'\Lambda(\Pi'_s - \mathbf{I})\Lambda^s(p_{.s}^+ - p_{.s}^-).$$

The left hand side of the inequality is greater than or equal to 0 whereas the right hand side is less than or equal to 0. The inequality turns into an equality and each summand on the right hand side has to equal 0. Or written differently

$$\sum_{j \in \mathcal{I}} \Theta_{ij} = 1 \quad \text{for all } i \in \mathcal{I}^0$$

and

$$\sum_{j \in \mathcal{I}} \Pi_{ijs} = 1 \quad \text{for all } i \in \mathcal{I}^s.$$

To prove the third claim note that

$$\Lambda V^+ = \Lambda(e + \sum_{s=1}^{S^*} \Pi'_s p_{.s}^+ - \sum_{s=1}^{S^*} p_{.s}^+ + \Theta' V^+).$$

Premultiplying by  $\vec{1}'$  yields

$$\vec{1}'\Lambda(\mathbf{I} - \Theta')V^+ = \vec{1}'\Lambda e + \sum_{s=1}^{S^*} \vec{1}'\Lambda(\Pi'_s - \mathbf{I})p_{.s}^+.$$

Banks with unique equity value but non-unique debt payments have to have an equity value of 0, i.e.  $(\Lambda - \Lambda^0)V^+ = \vec{0}$ . Together with  $\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda^0 V^+ = 0$  the left hand side equals  $\vec{1}'\Lambda(\mathbf{I} - \Theta')(\mathbf{I} - \Lambda)V^+ = -\vec{1}'\Lambda\Theta'(\mathbf{I} - \Lambda)V^+$ . Rearranging yields

$$\sum_{s=1}^{S^*} \vec{1}'\Lambda(\mathbf{I} - \Pi'_s)\Lambda p_{.s}^+ = \vec{1}'\Lambda e + \vec{1}'\Lambda\Theta'(\mathbf{I} - \Lambda)V^+ + \sum_{s=1}^{S^*} \vec{1}'\Lambda\Pi'_s(\mathbf{I} - \Lambda)p_{.s}^+$$

or

$$\sum_{s=1}^{S^*} \sum_{i \in \mathcal{I}} \left( 1 - \sum_{j \in \mathcal{I}} \Pi_{ijs} \right) p_{is}^+ = \sum_{i \in \mathcal{I}} e_i + \sum_{s=1}^{S^*} \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Pi_{ijs} \right) p_{is}^* + \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Theta_{ij} \right) V_i^*$$

Note that the right hand side does not depend on the chosen clearing vector as  $(\mathbf{I} - \Lambda)p_s^+ = (\mathbf{I} - \Lambda)p_s^-$  and  $(\mathbf{I} - \Lambda)V^+ = (\mathbf{I} - \Lambda)V^-$ . Assume that only one seniority class exists, i.e.  $S^* = \vec{1}$ . The  $i$ -th entry in the diagonal of  $(\Lambda - \Lambda^1)$  equals 1 if bank  $i$ 's debt payment is unique but the equity value is non-unique. In this case  $p_{i1}^+ = \bar{p}_{i1}$ , i.e.  $(\Lambda - \Lambda^1)p_{i1}^+ = (\Lambda - \Lambda^1)\bar{p}_{i1}$ . We get

$$\vec{1}'\Lambda(\mathbf{I} - \Pi_1')\Lambda\bar{p}_{i1} = \vec{1}'\Lambda e + \vec{1}'\Lambda\Theta'(\mathbf{I} - \Lambda)V^* + \vec{1}'\Lambda\Pi_1'(\mathbf{I} - \Lambda)p_{i1}^*$$

where  $V_i^* = V_i^+ = V_i^-$  and  $p_{i1}^* = p_{i1}^+ = p_{i1}^-$ . Alternatively, we may write

$$\sum_{i \in \mathcal{I}} \left( 1 - \sum_{j \in \mathcal{I}} \Pi_{ij1} \right) \bar{p}_{i1} = \sum_{i \in \mathcal{I}} e_i + \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Pi_{ij1} \right) p_{i1}^* + \sum_{i \notin \mathcal{I}} \left( \sum_{j \in \mathcal{I}} \Theta_{ij} \right) V_i^*.$$

$$\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda^0 = \vec{1}'\Lambda(\Pi_s' - \mathbf{I})\Lambda^s = 0 \quad \text{for all } s.$$

□

Start with  $p^0 = \bar{p}$  and calculate  $W^*(p^0)$  using

$$W^*(p) = e + \sum_{s=1}^{S^*} (\Pi_s)'p_{\cdot s} - \sum_{s=1}^{S^*} \bar{p}_{\cdot s} + \Theta(W^*(p) \vee \vec{0}).$$

Let

$$p_{iT}^k = \left\{ \left[ W_i^*(p^{k-1}) + \sum_{s=T}^{S^*} \bar{p}_{is} \right] \vee 0 \right\} \wedge \bar{p}_{iT}$$

and iterate the procedure.  $W^*(p)$  is increasing in  $p$  implying that  $p^k \leq p^{k-1}$  for all  $k$  and  $\lim_{k \rightarrow \infty} p^k = p^+$  where  $p^+$  denotes the largest clearing vector.

**Theorem 4.** If  $\Theta$  is a holding matrix, the sequence

$$p_{iT}^k = \left\{ \left[ W_i^*(p^{k-1}) + \sum_{s=T}^{S^*} \bar{p}_{is} \right] \vee 0 \right\} \wedge \bar{p}_{iT}$$

started at  $p^0 = \bar{p}$  is well defined, decreasing, and converges to the largest clearing vector

$p^+$ .

*Proof.*  $p^{k+1}$  is well defined if  $W^*(p^k)$  is well defined. This is the case as  $\Theta$  is a holding matrix.

To calculate  $W^*(p^k)$  let

$$u = e + \sum_{s=1}^{S^*} (\Pi_s)' p_{\cdot s}^k - \sum_{s=1}^{S^*} \bar{p}_{\cdot s},$$

$W^0 = u$ ,  $\Lambda^j = \text{diag}(W^j > \vec{0})$ , and  $W^{j+1} = u + \Theta' \Lambda^j W^{j+1}$ . As  $\Theta' \Lambda^j$  is a holding matrix Lemma 1 implies that  $W^{j+1}$  exists and is unique. By construction  $\Lambda^j W^j \geq \Lambda^{j-1} W^j$ . Therefore  $u + \Theta' \Lambda^j W^j \geq u + \Theta' \Lambda^{j-1} W^j = W^j$ . Let  $y = u + \Theta' \Lambda^j W^j - W^j \geq \vec{0}$ . It holds that  $W^{j+1} - W^j = y + \Theta' \Lambda^j (W^{j+1} - W^j)$ . Applying Lemma 3 implies that  $W^{j+1} - W^j \geq y \geq \vec{0}$ . This in turn implies that  $\Lambda^{j+1} \geq \Lambda^j$ . If  $\Lambda^j = \Lambda^{j-1}$ , it follows that  $W^{j+1} = W^j$  and  $\Lambda^j W^j = (W^j \vee \vec{0})$ . Hence,  $W^j$  is a solution to  $W = u + \Theta(W \vee \vec{0})$ . If  $\Lambda^j \neq \Lambda^{j-1}$ , the procedure has to be continued. The iteration has to stop after at most  $n$  steps because  $\Lambda^j \leq \mathbf{I}$  for all  $j$ .

To prove that  $p^k$  is decreasing note that  $p^1 \leq p^0 = \bar{p}$  by construction. Now suppose  $p^0 \geq p^1 \geq \dots \geq p^i$ .  $W^*(p)$  is increasing in  $p$ . Hence,  $W^*(p^k) \leq W^*(p^{k-1})$  and therefore  $p^{k+1} \leq p^k$ . Now suppose the series converges to some  $\tilde{p}$ . This implies that

$$\tilde{p}_{iT} = \left\{ \left[ W_i^*(\tilde{p}) + \sum_{s=T}^{S^*} \bar{p}_{is} \right] \vee 0 \right\} \wedge \bar{p}_{iT}$$

So  $\tilde{p}$  is a clearing vector. Next note that  $W^*(p^0) \geq W^*(p^+)$  where  $p^+$  is the largest clearing vector. This implies that  $p^1 \geq p^+$ . Now suppose it holds for  $k$  up to  $l$  that  $p^k \geq p^+$ . Hence,  $W^*(p^l) \geq W^*(p^+)$ . But this implies that  $p^{l+1} \geq p^+$  and  $\tilde{p} \geq p^+$ . As  $p^+$  is the largest clearing vector by assumption,  $\tilde{p} = p^+$ .  $\square$