

A New Keynesian Model with Heterogeneous Expectations*

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Preliminary Version, May 17, 2007

Abstract

This paper introduces heterogeneous expectations into a New Keynesian model. Our primary theoretical contribution is to provide an aggregation result for a model with nominal rigidities and heterogeneous bounded rationality. We incorporate bounded rationality at the individual agent level and aggregate to obtain the model's associated IS and AS relations. We find that the Phillips curve depends on current expectations of both contemporaneous and future endogenous state variables, and that the IS and AS relations naturally reduce to the usual New Keynesian model as all boundedly rational agents become rational. This reduction allows for explicit analysis of the impact of heterogeneity on the model's implied dynamics. As an example of this type of analysis, we consider a case in which a portion of agents are rational and the rest are simple adaptive, and we analyze the model's resulting determinacy properties. We show that incorporating heterogeneous expectations into the New Keynesian model may significantly alter these properties: models that are determinate under the assumption of rationality may possess multiple equilibria in the presence of expectations heterogeneity, even for small departures from rationality.

JEL Classifications: E52; E32; D83; D84

Key Words: Heterogeneous expectations, monetary policy, adaptive learning, aggregation.

*We are grateful to Chuck Carlstrom, Larry Christiano, George Evans, and Glenn Rudebusch for suggestions. We also thank Fabio Milani and seminar participants at San Francisco FRB and Cleveland FRB for comments.

1 Introduction

During the past decade the New Keynesian monetary model has become a benchmark laboratory for the analysis of monetary policy. Despite a literature that has coordinated on this model, there is a surprisingly diverse set of policy advice: some advocate simple, implementable policy rules (McCallum and Nelson (1999), Schmitt-Grohe and Uribe (2005)); optimal state-contingent monetary policy (Giannoni and Woodford (2002), Woodford (2003), Benigno and Woodford (2005)); or, inflation targeting or inflation forecast targeting (Svensson (1999), Svensson and Woodford (2003), Svensson and Williams (2005)). One theme emerging from this literature is that policy should account for its effect on private-sector expectations of future policy actions. Indeed, Woodford (2006) emphasizes the role policy inertia plays in influencing expectations. Even in the presence of non-rational agents, it is still advised that policy be set while remaining cognizant of the feedback onto beliefs, though often with the added condition that the associated rational expectations equilibrium (REE) is stable under learning (Bullard and Mitra (2002)).

The emergence of this theme stems from models that assume a representative agent structure and homogeneous private-sector expectations.¹ This assumption is widespread despite increasing evidence that agents have heterogeneous expectations. For example, Mankiw, Reis and Wolfers (2003) document disagreement among both consumers and professional economists in survey data on inflation expectations. Carroll (2003) shows that information slowly diffuses through the economy. Branch (2004, 2005) finds that survey data on inflation expectations are consistent with a dynamic discrete choice between statistical predictor functions. The dynamic effect of expectations heterogeneity, though, has not yet been studied in the context of micro-founded monetary models. This paper introduces heterogeneous expectations into a New Keynesian framework and studies its implications for the dynamic equilibrium properties of the economy.

This paper's principal contribution is the development of aggregate IS and AS relations that are derived from a micro-founded sticky price model and that are also consistent with heterogeneous bounded rationality. Our model's formulation is couched in a simple yeoman farmer framework along the lines of Woodford (2003); however, instead of assuming that the farmers form their expectations rationally, we impose bounded rationality by providing farmers with expectations operators that are not fully optimal, and we allow for the possibility that expectations operators may differ across groups of farmers. The farmers' choices are then modeled as being made optimally given their forecasts of the economy's current and future states.

The learning literature has examined a number of boundedly rational expecta-

¹Heterogeneous expectations between the policymakers and the private sector have been analyzed by Honkapohja and Mitra (2006) and Sargent (1999).

tions operators – for instance, naive and simple adaptive expectations as well as more sophisticated expectations based on least-squares or constant gain learning. In these types of models, expectations of private agents are assumed to satisfy a number of intuitive properties, including linearity and the law of iterated expectations (LIE). To remain agnostic as to the type of agents in our economy, we adopt an axiomatic approach: we characterize a class of admissible boundedly rational expectations operators by specifying six axioms that they are required to satisfy. These axioms incorporate the intuitive properties mentioned above as well as a few more stringent requirements required to obtain analytic aggregation, including the suspension of higher order beliefs and common expectations of forecasted limiting wealth differences. These axioms simultaneously allow for a tractable model of heterogeneous beliefs as well as emphasize the explicit and somewhat strong assumptions that underlie the aggregation result.

The resulting reduced form model is similar to the standard IS and AS relations except that conditional expectations are replaced by a general expectations operator that is a convex combination of boundedly rational expectations, and also that the aggregate supply equation conditions on current expectations of both future *and* contemporaneous endogenous state variables.² We further find that if a proportion α of agents are rational and if $\alpha \rightarrow 1$ then the reduced form model reduces to the usual New Keynesian IS and AS relations.³

The explicit reduced form model in case of heterogeneity in expectations, together with its ability to capture the rational model as a limiting case, allows for the analysis of the dynamic implications of expectations heterogeneity. As an example, we consider the simple case in which a proportion of agents are rational and the remaining form their expectations adaptively. We find that the impact of increasing the proportion of adaptive agents depends critically on how the adaptive predictor weighs past data. If past data is discounted, then heterogeneity may be stabilizing in the sense that policy rules that result in indeterminacy under full rationality may yield determinacy when even a small proportion of agents are adaptive. On the other hand, if agents place greater weight on past data, then heterogeneity may be destabilizing in an analogous sense: policy rules yielding determinacy under rational expectations may produce indeterminacy if even a small proportion of agents are adaptive.

The heterogeneous expectations framework facilitates a generalization of the manner in which policy affects private-sector expectations. Importantly, monetary policy influences expectations of current and future policy decisions. The conduct of sound policy depends on the distribution of heterogeneity since heterogeneous expectations

²Gali, Lopez-Sallido, and Valles (2004) develop a model with rational and myopic agents. The heterogeneity is in consumption-saving behavior rather than expectations. In our model, all agents solve the same problem, have the same utility function, and face the same constraints. They differ, however, in how they evaluate their utility flows, which may imply different individual wealth levels.

³We must be careful when defining “rationality” in this model: see Section 3 below.

can alter the way in which expectations propagate shocks. This reasoning ties in with policymakers' frequently stated priority to prevent inflation expectations from becoming unhinged (e.g. Bernanke (2004)). In the current framework, how aggressive monetary policy needs to be in responding to inflation depends on the distribution of heterogeneity and how strongly boundedly rational agents respond to past data.

This paper is organized as follows. Section 2 presents the axioms that generalizes the New Keynesian model to include heterogeneous expectations. Section 3 presents an example with an explicit specification of heterogeneity to facilitate analysis of the model's equilibrium properties. Section 4 concludes.

2 Heterogeneous Bounded Rationality

This section develops a version of the New Keynesian model extended to include two types of agents who are identical except with respect to the method used to form expectations. We axiomatize expectations operators in a manner that will still allow for aggregation across heterogeneous agents. Our primary finding is an aggregation result: we derive log-linearized IS and AS relations that depend on a convex combination of heterogeneous expectations operators. To ease exposition, we assume a purely forward-looking specification of the model, though it would be straightforward to extend our methodology to a version of the model that incorporates habit persistence and partial price indexation (e.g. Christiano, Eichenbaum, and Evans (2005)).

2.1 Heterogeneous Expectations: Axioms

Following Woodford (2003) there is a continuum of private agents indexed by $i \in [0, 1]$. These agents are yeoman farmers with linear production functions $Y^i = N^i$, where agent i produces good i by supplying labor N^i . Agents maximize the expected value of discounted utility flow, but instead of assuming that agents form expectations rationally, we allow for more general expectations operators. We assume that a proportion α of agents forecast future variables using the expectations operator E^1 and the remaining agents use E^2 , and for simplicity, we assume an agent producing good $i \in [0, \alpha]$ uses E^1 and an agent producing good $j \in (\alpha, 1]$ uses E^2 . We now develop a framework in which equilibrium inflation and output depend on a heterogeneous expectations operator \hat{E}_t that is a linear combination of E^1, E^2 : $\hat{E} = \alpha E^1 + (1 - \alpha) E^2$.

Our view of expectation formation is directed by the adaptive learning literature (e.g. Marcet and Sargent (1989), Evans and Honkapohja (2001)). In this literature, fully rational expectations are replaced by linear forecasting rules whose parameters are updated by recursive least squares. We also envision agents engaged in economic forecasting while recognizing there may exist heterogeneity in forecasting rules. Some

examples of heterogeneity consistent with our framework include the following: some agents may be rational while others adaptive, as has been examined in a cobweb model by Brock and Hommes (1997) and found empirically relevant in the data in Branch (2004); agents may have different information sets (e.g. Mankiw, Reis, and Wolfers (2003)); or, they may use structurally different learning rules as in Honkapohja and Mitra (2005). Our goal is to extend this notion of agents as forecasters to the agents' primitive problem, and we begin by formalize the properties our expectations operators must have.

Denote by E_t^τ a given expectations operator; that is, we write $E_t^\tau(x_{t+k})$ for the time t expectation of x_{t+k} as formed by an agent of type τ . We impose that

- A1. If x is a variable forecasted by agents and has steady state \bar{x} then $E^1\bar{x} = E^2\bar{x} = \bar{x}$.
- A2. If x , y , $x + y$ and αx are variables forecasted by agents then $E_t^\tau(x + y) = E_t^\tau(x) + E_t^\tau(y)$ and $E_t^\tau(\alpha x) = \alpha E_t^\tau(x)$.
- A3. If for all k , x_{t+k} and $\sum_k \beta^{t+k} x_{t+k}$ are forecasted by agents then

$$E_t^\tau \left(\sum_k \beta^{t+k} x_{t+k} \right) = \sum_k \beta^{t+k} E_t^\tau(x_{t+k}).$$

- A4. E_t^τ satisfies the law of iterated expectations (L.I.E.): If x is a variable forecasted by agents at time t and time $t + k$ then $E_t^\tau \circ E_{t+k}^\tau(x) = E_t^\tau(x)$.
- A5. If x is a variable forecasted by agents at time t and time $t+k$ then $E_t^\tau E_{t+k}^{\tau'}(x_{t+k}) = E_t^\tau x_{t+k}$, $\tau' \neq \tau$.
- A6. All agents have common expectations on expected differences in wealth.

These assumptions serve two purposes: they impose regularity on the expectations operator consistent with the literature on bounded rationality, and they facilitate aggregation. Because these restrictions are necessary for aggregation, the following brief discussion of each assumption is useful. Assumption A1 requires some continuity in beliefs in the sense that, in a steady state, agents' beliefs will coincide. Assumptions A2 and A3 require expectations to possess some linearity properties. Essentially, linear expectations require agents to incorporate some economic structure into their forecasting model rather than, say, mechanically applying a lag operator to every random variable.

Assumptions A4 and A5 restrict agents' expectations so that they satisfy the law of iterated expectations at both an individual and aggregate level. The L.I.E. at the individual level is a reasonable and intuitive assumption: agents should not expect

to systematically change their expectations. Assumption A5 is more subtle and is necessary so that aggregate expectations satisfy the law of iterated expectations. The assumption of L.I.E. at the aggregate level is not without consequence, however: A5 requires that an agent of type τ 's expectation of the future expectations of agents of type τ' is τ 's expectation of that future variable. This assumes away higher-order beliefs of boundedly rational agents. There is a wide literature that studies the implications of higher-order beliefs in monetary models (e.g. Woodford (2002), Amato and Shin (2006)). Higher order beliefs may be important in some settings and its consequence for economic dynamics is a topic that may warrant further study.

Assumption A6 (made more precise below) is perhaps the most restrictive assumption. In the formulation of expectations there will be wealth dynamics that differ by expectations type. These wealth dynamics, at first glance, appear to cause a problem for aggregation: if wealth dynamics matter for aggregate variables then it will be necessary to keep track of the wealth distribution and incorporate its evolution into the reduced form model. The axiom that agents agree on limiting wealth distributions avoids this added difficulty thereby allowing us to remain close in form to the homogeneous case.

2.2 Households

Agents have instantaneous utility functions given by $u\left(C_t^i, \frac{M_t^i}{P_t}\right) - v(N_t^i)$, where U is separable and C^i, P are the CES aggregators

$$C^i = \left(\int C^i(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} \quad \text{and} \quad P = \left(\int P(j)^{1-\theta} dj \right)^{\frac{1}{1-\theta}}. \quad (1)$$

Agents choose sequences for consumption, money holdings, labor, and goods prices to maximize expected discounted utility subject to a flow constraint. However, as is standard in the bounded rationality and learning literature, agents do not necessarily observe current values of endogenous variables when making decisions: it may be that $E_t^\tau x_t \neq x_t$. Quite naturally, we do assume that current values of an individual agent's own choice variables are known.

Under rationality, the agent's contingency plans for the consumption bundle, prices, labor, bonds, and money holdings are chosen to satisfy certain familiar intertemporal first order conditions, which will depend in part on conditional expectations formed with respect to the true distributions of the relevant variables. Under bounded rationality, the optimal solution is more subtle. For general subjective expectations E_t^i it is not obvious that the analogous first order conditions are necessary or sufficient for the contingency plans to be optimal: the theory of dynamic programming and the principle of optimality were developed in stochastic environments assuming E_t^i are conditional expectations, and in the present setting, it is not clear

that the relevant theorems apply.⁴ Instead, we follow the learning literature by taking as primitive that agents with subjective (non-rational) expectations choose plans that satisfy the associated Euler equations. And the justification for this primitive assumption is derived directly from the variational intuition for the usual Euler equations in case of rationality: we simply assume that agents make choices by equating expected marginal benefit with expected marginal cost.

Agents choose consumption bundles to satisfy an intertemporal Euler equation of the form

$$u_c(C_t^i) = \beta(1 + i_t)E_t^i \left(\frac{P_t}{P_{t+1}} \right) u_c(C_{t+1}^i). \quad (2)$$

Intuitively, the consumption bundle choice must be optimal with respect to the agent's time t perceived trade-off. In the literature on bounded rationality, this interpretation is called Euler equation learning. Preston (2006) presents an interesting alternative approach, in a representative agent environment, where the agents' optimal plans also respect their perceived lifetime *intertemporal* budget constraint.⁵ Explicitly incorporating long-horizon expectations into the model is the next step and is a topic of current research.

We follow Woodford (2003) in assuming the cashless limit case: the utility function is such that each household holds an arbitrarily small amount of real-money so that the central bank can effectively choose a target level for the interest rate and, although the real-money demand function dictates the size of open market operations consistent with that target, the actual effect of these operations on the net supply of bonds is negligible. In the current environment, the money-demand equation is given by

$$\frac{u_M(C_t^i, M_t^i/P_t)}{u_C(C_t^i, M_t^i/P_t)} = \frac{i_t}{1 + i_t}. \quad (4)$$

Notice as well that the expression for money demand (4) does not depend explicitly on agents' expectations.⁶

We adopt the Calvo pricing structure, where with a positive probability each farmer's price may remain fixed. Acting as price setters, individual agents face the risk associated with this Calvo lottery.⁷ The standard treatment allows agents to

⁴We have not assumed that E_t^i may be thought of as an mathematical expectation with subjective prior.

⁵As is standard in Euler equation learning, we do require that the agent's subjective transversality condition, given by

$$\lim_{k \rightarrow \infty} E_t^i \beta^{t+k} u_C(C_{t+k}^i) \frac{B_{t+k}^i}{P_{t+k}} \leq 0. \quad (3)$$

is satisfied ex-post. See Evans, Honkapohja, and Mitra (2004) for a discussion.

⁶In incomplete market settings i_t may depend on the distribution of agents' wealth holdings (e.g. Krusell and Smith (1998), Aiyagari (1994, 1995)). Our modeling of risk-sharing below makes wealth holdings a fraction of the real-income of their type cohort.

⁷For the purposes of this paper, we abstract from modeling aggregate risk.

hedge against this risk by assuming the presence of complete markets: in this way, agents are able to guarantee themselves the average real income each period. Our model posits two types of agents. While each type will face an equal probability of being allowed to change prices, their expected real income differs due to heterogeneity of expectations. Because of this, deriving from micro-foundations a reduced form requires some care. For example, the derivation of the IS relation, which is obtained by log-linearizing the agents' consumption Euler equation, requires agents of a given type form expectations of their own consumption index, and the value of this index may differ from the index value of other types of agents; it is not immediate that aggregation across types is possible in this case.

To address this issue, we assume the presence of a benevolent financial regulator who takes insurance premiums optimally from each type of consumer and then returns the proceeds in order to provide agents the means to hedge the risk associated to the Calvo lottery. The planner collects all income and then redistributes to each type of agent the average income of that agent's type. The planner does this so that each agent type, as indexed by their expectation formation mechanism, is fully insured against the risk associated to the possibility that they will not be tapped to adjust prices; in this way agents are able to guarantee themselves within type average real income. This approach is similar to Kocherlakota (1996) except we assume there are no commitment issues, on either side, and this essentially creates two types of representative agents. This risk-sharing mechanism is also employed by Shi (1999) and Mankiw and Reis (2006).

As an alternative there is a long history of studying heterogeneity in dynamic models with incomplete financial markets. In many cases, the existence of heterogeneous consumers and incomplete markets imply that equilibrium prices and allocations differ from the representative agent model, but the quantitative differences seem small (Krusell and Smith (1998)). In our setting, heterogeneity arises because agents use different expectations operators when solving for their optimal plans, thus their optimal allocations could be different. Our intent is to provide an axiomatic foundation for the model with heterogeneous expectations to remain close, in reduced form, to the standard model.

Under our financial structure, agents of type τ may guarantee themselves a real income Ω^τ as given by

$$\Omega^1 = \frac{1}{\alpha P} \int_0^\alpha P(i)Y(i)di \quad \text{and} \quad \Omega^2 = \frac{1}{(1-\alpha)P} \int_\alpha^1 P(i)Y(i)di. \quad (5)$$

Notice that

$$P(\alpha\Omega^1 + (1-\alpha)\Omega^2) = \int P(i)Y(i)di = PY,$$

from which it follows that $Y = \alpha\Omega^1 + (1-\alpha)\Omega^2$.

In equilibrium, agents choices must satisfy their (ex-post) nominal budget constraint:

$$C_t^i P_t + M_t^i + B_t^i + I_{pt} = P_t^i Y_t^i + (1 + i_{t-1}) B_{t-1}^i + M_{t-1}^i + I_{rt}, \quad (6)$$

where P^i is the price set by agent i , Y^i is the quantity of goods produced by agent i , M^i is the money held by agent i , B^i are the bonds held by agent i , and I_p and I_r are nominal payments to and receipts from an insurance agency respectively, which the agent takes as given. If agent i is of type τ then, $I_{rt}^i = P_t \Omega_t^\tau$ and $I_{pt}^i = P_t^i Y_t^i$. Therefore, in equilibrium, the real budget constraint requires

$$C_t^i + b_t^i = \left(\frac{1 + i_{t-1}}{1 + \pi_t} \right) b_{t-1}^i + \Omega_t^\tau, \quad (7)$$

where b^i is the quantity of real bonds held by agent i , and $1 + \pi_t = P_t/P_{t-1}$. A first order expansion of (7) around steady state, incorporating A1, yields

$$C_t^i - \bar{C}^i + b_t^i = \left(\frac{1 + \bar{i}}{1 + \bar{\pi}} \right) b_{t-1}^i + \Omega_t^\tau - \bar{\Omega}^\tau,$$

where we have exploited $\bar{b}^i = 0$. Using $\beta(1 + \bar{i}) = 1$ and $\bar{C}^i = \bar{\Omega}^\tau = \bar{Y}$, we obtain the log-linear approximation

$$c_t^i = \hat{\Omega}_t^\tau \equiv \omega_t^\tau + \beta^{-1} \frac{b_{t-1}^i}{\bar{Y}} - \frac{b_t^i}{\bar{Y}}, \quad (8)$$

where c and ω are in log deviations from steady-state form, $c = \log(C^i/\bar{Y})$, $\omega = \log(\Omega^\tau/\bar{\Omega}^\tau)$.

To obtain a New Keynesian IS relation, we log-linearize (2):

$$c_t^i = E_t^i c_{t+1}^i - \sigma^{-1} (i_t - E_t^i \pi_{t+1}). \quad (9)$$

We note that $E_t^i \pi_{t+1} = E_t^i (\log P_{t+1} - \log P_t) = E_t^i \log P_{t+1} - E_t^i \log P_t$, and may not be equal to $E_t^i (\log P_{t+1}) - \log P_t$, in the event P_t is not observable.

Inserting (8) into (9) yields the equation

$$\hat{\Omega}_t^\tau = E_t^\tau \hat{\Omega}_{t+1}^\tau - \sigma^{-1} (i_t - E_t^\tau \pi_{t+1}). \quad (10)$$

Note that along any equilibrium path, equation (10) must be satisfied for both agent types. The IS curve comes from aggregating this equation across all households. Iterate this equation forward, while employing the assumptions on agents' expectations, to obtain

$$\hat{\Omega}_t^\tau = \hat{\Omega}_\infty^\tau - \sigma^{-1} E_t^\tau \sum_{k \geq 0} (i_{t+k} - \pi_{t+k+1}), \quad (11)$$

where $\hat{\Omega}_\infty^\tau = \lim_{k \rightarrow \infty} E_t^\tau \hat{\Omega}_{t+k}^\tau$.⁸ Noting that bond market clearing requires $\alpha b_t^1 = -(1 - \alpha)b_t^2$, we get that

$$\alpha \hat{\Omega}_t^1 + (1 - \alpha) \hat{\Omega}_t^2 = y_t.$$

Then

$$\begin{aligned} y_t &= \alpha \hat{\Omega}_t^1 + (1 - \alpha) \hat{\Omega}_t^2 \\ &= \alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2 - \sigma^{-1} \hat{E}_t \sum_{k \geq 0} (i_{t+k} - \pi_{t+k+1}) \\ &= \hat{E}_t y_{t+1} - \sigma^{-1} (i_t - \hat{E}_t \pi_{t+1}) \\ &\quad + \alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2 - \hat{E}_t (\alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2), \end{aligned}$$

where $\hat{E} = \alpha E_1 + (1 - \alpha) E_2$. Here we note that the final equality makes use of the law of iterated expectations at the aggregate level: see assumption (A5).

To obtain an IS form most similar to the homogeneous expectations case, we now employ A6, the assumption that all agents predict limiting wealth in an identical manner:

$$\alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2 - \hat{E}_t (\alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2) = 0.$$

Assumption (A6) can be written equivalently as $\hat{\Omega}_\infty^j = E^{j'} \hat{\Omega}_\infty^j, j' \neq j$. While this is plausible, and is satisfied when agents have rational expectations regarding limiting wealth, one might envision agents holding beliefs that violate it. In that case, the distribution of wealth dynamics will affect equilibrium allocations. This axiom sheds light on the limitations of a reduced-form theory of heterogeneous expectations.

The above analysis provides the following proposition:

Proposition 1 *If agents' expectations E^1 and E^2 satisfy assumptions (A1)-(A6) then, up to a log-linear approximation, equilibrium output and inflation satisfy the following IS relation*

$$y_t = \hat{E}_t y_{t+1} - \sigma^{-1} (i_t - \hat{E}_t \pi_{t+1}) \quad (12)$$

where $\hat{E} = \alpha E^1 + (1 - \alpha) E^2$.

From this we conclude that with heterogeneous expectations the IS equation is the usual IS curve where conditional expectations are replaced with a convex combination of the heterogeneous expectations operators.

⁸We take the random variable $\hat{\Omega}_\infty^\tau$, which is the time t expected limiting value of log consumption, to be finite and independent of time.

2.3 Optimal Pricing

To determine the equilibrium behavior of inflation we turn to the agents' pricing problem. We first compute demand for a particular good: the CES aggregate consumption index implies that demand by an agent of type τ for good i , is given by $(P_i/P)^{-\theta} (\Omega^\tau + (1 + i_{-1})B_{-1}^\tau/P - B^\tau/P)$. Aggregating across type and imposing bond market clearing yields

$$Y^i = \alpha (P_i/P)^{-\theta} \Omega^1 + (1 - \alpha) (P_i/P)^{-\theta} \Omega^2 = (P_i/P)^{-\theta} Y$$

as usual.

Note that because of the presence of the insurance company, an agent's income is, in effect, independent of his effort; because of this, an agent's optimal output Y^i is zero. To avoid this free-rider problem, we assume that private agents are contracted to choose price and output as if they faced their perceived trade-off, and we further assume that these contracts are fully enforceable. In some sense, this requires the agent to behave as if they will receive their full marginal revenue from producing more.⁹ With this assumption, we may proceed as follows: Let agent i be of type τ . Let $C_{t+k}^i = C_{t+k}^i(P_t^i)$ be the consumption bundle in the event that agent i can not change prices for k periods. Then P_t^i is chosen by contract to solve

$$\max E_t^\tau \sum_{k \geq 0} (\beta\gamma)^k \left[u(C_{t+k}^i(P_t^i), \cdot) - v\left(\left(P_t^i/P_{t+k}\right)^{-\theta} Y_{t+k}\right) \right]. \quad (13)$$

Our implementation of bounded rationality requires that agents choose prices to satisfy the first order conditions of this pricing problem, with subjective expectations replacing rational ones. This FOC may be log-linearized as usual, but now there is the presence of both Y and C^i . Passing expectations across choice variables (so that $\hat{\Omega}_t^\tau$ and not $E_t^\tau \hat{\Omega}_t^\tau$ enters the equation) and imposing equilibrium yields

$$E_t^\tau \sum_{k \geq 0} (\gamma\beta)^k \left(\log(P_t^i) - \log(P_{t+k}) - \zeta_1 \hat{\Omega}_{t+k}^\tau - \zeta_2 y_{t+k} \right) = 0.$$

Subtract $E_t^\tau \log P_t$ from each side and solve for $\log P_t^i - E_t^\tau \log P_t$. We obtain¹⁰

$$\log P_t^i - E_t^\tau \log P_t = E_t^\tau \sum_{k \geq 0} (\gamma\beta)^k \left(\gamma\beta \pi_{t+k+1} + (1 - \gamma\beta)(\zeta_1 \hat{\Omega}_{t+k}^\tau + \zeta_2 y_{t+k}) \right).$$

⁹One could equivalently imagine a household where one member consumes given the insurance payment and the other works and makes optimal pricing decisions.

¹⁰This step requires the following observation: Let $s_t = \sum_{k=1}^t \pi_k = \log P_t - \log P_0$, and set $s_0 = 0$. Then

$$(1 - \delta) \sum_{t \geq 0} \delta^t (\log P_t - \log P_0) = \sum_{t \geq 0} \delta^t s_t - \sum_{t \geq 0} \delta^{t+1} s_t = \sum_{t \geq 0} \delta^t \pi_{t+1}.$$

Stepping this equation forward, applying the L.I.E., using the linearity assumption (A2) and rearranging yields

$$\log P_t^i - E_t^\tau \log P_t = \gamma\beta E_t^\tau \pi_{t+1} + (1-\gamma\beta) \left(\zeta_1 \hat{\Omega}_t^\tau + \zeta_2 E_t^\tau y_t \right) + \gamma\beta E_t^\tau \log P_{t+1}^i / P_{t+1}. \quad (14)$$

We turn now to the evolution of aggregate prices. The pricing decision is homogeneous within type, so we now say P_t^τ is the optimal price chosen by an agent of type τ in time t , and $P_t^\tau(j)$ denotes the price set by firm j (of type τ) in time t . Then

$$(P_t)^{1-\theta} = \int_0^\alpha P_t^1(j)^{1-\theta} dj + \int_\alpha^1 P_t^2(j)^{1-\theta} dj.$$

The proportion $1 - \gamma$ of each type changes their optimal price in period t . Thus

$$\int_0^\alpha P_t^1(j)^{1-\theta} dj = (1-\gamma)\alpha (P_t^1)^{1-\theta} + \gamma \int_0^\alpha P_{t-1}^1(j)^{1-\theta} dj,$$

and similarly for $\int_\alpha^1 P_t^2(j)^{1-\theta} dj$. We obtain

$$(P_t)^{1-\theta} = (1-\gamma)\alpha (P_t^1)^{1-\theta} + (1-\gamma)(1-\alpha) (P_t^2)^{1-\theta} + \gamma (P_{t-1})^{1-\theta}.$$

Log linearization yields

$$p_t = (1-\gamma)\alpha p_t^1 + (1-\gamma)(1-\alpha)p_t^2 + \gamma p_{t-1},$$

so that, subtracting $(1-\gamma)p_t$ from both sides, we have

$$\alpha \log P_t^1 / P_t + (1-\alpha) \log P_t^2 / P_t = \frac{\gamma}{1-\gamma} \pi_t.$$

Finally, multiply equation (14) by α for $\tau = 1$ and by $1 - \alpha$ for $\tau = 2$ and add to get the following result:

Proposition 2 *If agents' expectations E^1 and E^2 satisfy assumptions (A1)-(A6) then, up to a log-linear approximation, equilibrium output and inflation satisfies the AS relation*

$$\pi_t = \beta \hat{E}_t \pi_{t+1} + \lambda_1 y_t + \lambda_2 \hat{E}_t y_t + \frac{1-\gamma}{\gamma} (\hat{E}_t \pi_t - \pi_t). \quad (15)$$

where $\hat{E} = \alpha E^1 + (1-\alpha)E^2$.

Here $\lambda = \lambda_1 + \lambda_2$ is the usual coefficient on output gap in the new Keynesian Phillips curve. Notice that if all agents are rational, so that $\hat{E}_t = E_t$, this curve reduces to the usual AS equation.

It is worth remarking on an important difference between (15) and the usual New Keynesian expectational Phillips curve. Equation (15) includes two terms that incorporate current expectations of the current state variables, $\hat{E}_t y_t, \hat{E}_t \pi_t$. These terms arise because of the natural timing assumptions for boundedly rational agents. We assumed, as is standard in the adaptive learning literature, that boundedly rational agents observe current exogenous variables but contemporaneous endogenous state variables are unobserved. In price-setting this implies that agents do not see the aggregate price when they set their own price, and similarly for observing aggregate output. This assumption is often employed for adaptive agents to avoid simultaneity in beliefs and outcomes that, while natural in a rational expectations equilibrium, seems less plausible for boundedly rational agents. If these state variables were observed then the final term in (15) would reduce to zero and $\lambda_1 y_t + \lambda_2 \hat{E}_t y_t = \lambda y_t$, and we would have a heterogeneous expectations Phillips curve with the same reduced form as under RE. We anticipate that the form of (15) may have implications for an estimated version of the model and any potential empirical implications are the topic of future research.

2.4 Heterogeneous Expectations Equilibria

The aggregate dynamics of our model are given by

$$\begin{aligned} y_t &= \hat{E}_t y_{t+1} - \sigma^{-1} \left(i_t - \hat{E}_t \pi_{t+1} \right) \\ \pi_t &= \beta \hat{E}_t \pi_{t+1} + \lambda_1 y_t + \lambda_2 \hat{E}_t y_t + \frac{1-\gamma}{\gamma} (\hat{E}_t \pi_t - \pi_t), \end{aligned}$$

and it is tempting to think that, given a policy process i_t , an output gap and inflation process y_t, π_t satisfying the above system constitutes an equilibrium to our model; however, we must proceed with care to ensure that all of the assumptions regarding optimal agent behavior are respected. The definition of equilibrium must include the restriction that $\hat{\Omega}_t^\tau$ satisfies the Euler equation (10) for each agent type τ , and further that the transversality condition of each agent type is respected. We return to these issues in Section 3 below where, for our application to monetary policy and determinacy, we explicitly define the expectations operators.

2.5 Further Discussion

The above analysis demonstrates the conditions under which aggregation obtains in a New Keynesian model with heterogeneous expectations. The results demonstrate that, under the maintained axioms on expectations, individual decision rules aggregate into heterogeneous expectations IS and AS relations. The heterogeneous expectations Phillips curve incorporates current expectations of contemporaneous output

and inflation. These terms arise if some agents form decisions based on forecasts of contemporaneous endogenous state variables.

There are many simplifications in the standard New Keynesian model that facilitate representing the equilibrium in terms of reduced-form IS-AS equations. In this paper, we relax one assumption: that expectations are rational. We then ask under what conditions can the equilibrium still be written in a reduced-form in the presence of agents that do not hold rational expectations and in the event agents have heterogeneous expectations? The benefit of an axiomatic approach is that it makes these foundations clear. This subsection discusses at greater length some of the assumptions required for aggregation.

2.5.1 The Law of Iterated Expectations

The aggregation results, captured in Propositions 1 and 2, rely heavily on the law of iterated expectations, both at the individual level (assumption (A4)) and at the aggregate level (assumption (A5)), and some discussion of the application of this law is warranted. The law of iterated expectations at the individual level is a consequence of consistent behavior on the part of agents: private agents should not be able to forecast changes in future forecasts. Fortunately for the modeler, incorporation of LIE at the agent level is straightforward: expectations may be constructed recursively by forward iteration on the relevant forecasting models. This is the standard assumption in the literature on learning in which expectations of future variables are based on forward iteration of the agents' perceived laws of motion: see for example, Bullard and Mitra (2002) and Evans and Honkapohja (2003). Also, it is precisely this construction that we adopt in the next section when imposing adaptive expectations on a proportion of agents.

While we contend that the law of iterated expectations is a defining property of expectation formation at the individual level, imposing it at the aggregate level is more restrictive: assumption A5 is required and it rules out higher order beliefs. With respect to this property, models with expectations homogeneity have it easy – they get aggregate LIE for free. To obtain a tractable aggregation result, we impose A5, but note that if one imagines that the global interactions identified by higher order beliefs are important, then the aggregate dynamics for inflation and output will be altered.

3 An Application to Monetary Policy

In this section we investigate the aggregate model's determinacy properties. We anticipate that there are other applications of our framework but focus on a partic-

ular example that demonstrates the potential for heterogeneity to alter the dynamic properties of the economy.

A linear rational expectations model is said to be determinate if there is a unique non-explosive equilibrium, indeterminate if there are many non-explosive equilibria, and explosive otherwise, and we simply extend these definitions to the model with heterogeneous expectations.¹¹ That New Keynesian models, when closed with Taylor-type instrument rules, may exhibit indeterminacy has been noted by a number of authors: see, for example, Bernanke and Woodford (1997), Woodford (1999) and Svensson and Woodford (2003). Indeterminacy may be undesirable because of the presence of multiple equilibria, many of which may be welfare reducing. Clarida, Gali and Gertler (2000) have suggested that the volatile nature of the US time-series in the 1970s can, in part, be explained by the presence of sunspot equilibria. Further empirical evidence is provided by Lubik and Schorfheide (2005).

In a univariate reduced form model having the same structure as the heterogeneous NK model – see (21) below – Branch and McGough (2005) found that the presence of adaptive agents impacts the determinacy properties of the model. For some parameterizations, the presence of adaptive agents may be stabilizing in the sense that a model that is indeterminate under the assumption of uniform rationality may be determinate when some agents form their expectations adaptively. The converse also holds: for some parameterizations even if only a very small proportion of agents are adaptive, an otherwise determinate model may be indeterminate.

The importance from a policy perspective of understanding a monetary model's determinacy properties, together with the possibility demonstrated by Branch and McGough (2005) that the presence of adaptive agents can impact determinacy, motivates our analysis of the determinacy properties of the New Keynesian model with heterogeneous expectations. The presence of heterogeneous agents raises an obvious question: how are the determinacy properties of a such a model computed? This question was answered in Branch and McGough (2005): specifically, given a monetary policy $\{i_t\}$, and a pair of expectations operators (E^1, E^2) , solutions to (12), (15) are what Branch and McGough (2005) define as *Heterogeneous Expectations Equilibria*. In this earlier paper, we showed that any model with heterogeneous expectations of a particular form can be re-written in terms of an associated rational expectations model, bringing to bear the standard rational expectations toolkit to analyze the number and nature of heterogeneous expectations equilibria. In this way, the analysis of determinacy in our model reduces to the well-understood determinacy analysis of the associated rational expectations model.

Before analyzing determinacy, we close the heterogeneous expectations New Keynesian model by specifying a monetary policy rule and a precise form for private

¹¹Non-explosive equilibria are the focus of rational expectations model because explosive time series typically violate transversality.

agents' expectations.

3.1 Monetary Policy

The central bank is assumed to use a forward-looking Taylor rule to set interest rates:

$$\text{PR}_1 : i_t = \alpha_y E_t y_{t+1} + \alpha_\pi E_t \pi_{t+1}, \quad (16)$$

where expectations here are taken to be rational. This expectations based policy rule has been studied by Bullard and Mitra (2002), Evans and Honkapohja (2003a,b), Evans and McGough (2005a,b), and Preston (2005). It has the advantage of conditioning on observables, and thus addresses the concern of McCallum that contemporaneous policy rules are not realistic in practice; and it imposes that policy-makers set their instrument conditional on their best forecasts of future inflation and output gap (e.g. Svensson and Woodford (2003)): in this way, policy makers can account for the lag between a policy's implementation and its impact.

We adopt (16) as our benchmark policy rule; however, we also consider the following specification of the policy rule:

$$\text{PR}_2 : i_t = \alpha_y \hat{E}_t y_{t+1} + \alpha_\pi \hat{E}_t \pi_{t+1}. \quad (17)$$

Here, policy makers react to private agents expectations rather than to optimal forecasts; this is consistent with the view that policy-makers set policy to anchor agents' expectations. In particular, Bernanke (2004) advocates policy that reacts aggressively to private-sector expectations, particularly when they are non-rational. Evans and Honkapohja (2003) show that such a rule may have good properties in the presence of boundedly rational agents.

3.2 Rational versus Adaptive Expectations

In order to present results concerning our model's determinacy properties, we now make specific assumptions on \hat{E}_t . For simplicity, we focus on rational versus simple adaptive expectations. More specifically, we make the following assumption on beliefs: agents of type 1 are rational, and agents of type 2 are adaptive on output and inflation: thus for $x = y$ or π ,

$$E_t^2(x_t) = \theta_x x_{t-1},$$

and, as is standard in the learning literature, $E_t^2 x_{t+k}$ is constructed iteratively. Having defined adaptive expectations on x_{t+k} , we impose that the operator satisfy the assumptions (A1)-(A6). Note that from our construction, adaptive expectations satisfies the L.I.E. However, somewhat more subtly, rational agents are not "hyper-

rational” in the following sense: hyper-rationality would imply that $E_t^1 E_{t+1}^2 x_{t+1} = \theta_x x_t$; however, assumption (A5) requires that $E_t^1 E_{t+1}^2 x_{t+1} = E_t x_{t+1}$.¹²

Using (A5) we obtain

$$\begin{aligned}\hat{E}_t x_t &= n_x x_t + (1 - n_x) \theta_x x_{t-1} \\ \hat{E}_t(x_{t+1}) &= n_x E_t x_{t+1} + (1 - n_x) \theta_x^2 x_{t-1}.\end{aligned}$$

Here n_x is the proportion of agents having rational expectations. The operator E_t^2 is a form of adaptive ($\theta < 1$) or extrapolative ($\theta > 1$) expectations (for simplicity, below we refer to E^2 as an “adaptive” expectations operator, even in the event $\theta > 1$). Such expectations can be thought of as arising from a simple linear perceived law of motion $x_t = \theta x_{t-1}$. Under certain conditions, including the assumption of homogeneous expectations, it is possible for real-time estimates of θ to converge to their REE value. In this paper, we take the value of θ as given and leave to future research the study of stability under learning of heterogeneous expectations equilibria.¹³ We do, however, let the fraction of rational agents and the adaptive parameter θ differ by forecasting variable.

Adaptive expectations of this form have been considered by Brock and Hommes (1997, 1998), Branch (2002), Branch and McGough (2005), and Pesaran (1987). When $\theta = 1$ then the operator is usually called ‘naive’ expectations, $\theta < 1$ are adaptive in the sense that they dampen recent observations, $\theta > 1$ are often called ‘extrapolative’ or trend chasing expectations. The $\theta > 1$ is given particular emphasis by Brock and Hommes (1998).

Despite the simple form of our heterogeneous expectations operator, there is evidence in survey data on inflation expectations that agents are distributed across rational expectations and adaptive expectations of this form (see Branch (2004)). The contribution of this paper, however, is to show how to incorporate heterogeneous expectations into a monetary model and show that their presence can have significant implications for monetary policy. Future research should undertake serious empirical studies of heterogeneous beliefs in monetary DSGE models along the lines of Smets and Wouters (2003), Lubik and Schorfheide (2005), or Milani (2006).

¹²More carefully, our rational agents are assumed to know the conditional distributions of output and inflation, but not the expectations of non-rational agents. We further assume that rational agents are not sophisticated enough to back out the expectations of non-rational agents.

¹³Some progress on this issue has been made by Guse (2005) in a simple univariate framework.

3.3 Determining Determinacy

Our fully specified aggregate model is given by

$$y_t = \hat{E}_t y_{t+1} - \sigma^{-1} \left(i_t - \hat{E}_t \pi_{t+1} \right) \quad (18)$$

$$\pi_t = \beta \hat{E}_t \pi_{t+1} + \lambda_1 y_t + \lambda_2 \hat{E}_t y_t + \frac{1-\gamma}{\gamma} (\hat{E}_t \pi_t - \pi_t) \quad (19)$$

$$i_t = \alpha_y E_t y_{t+1} + \alpha_\pi E_t \pi_{t+1} \quad (20)$$

$$\hat{E}_t y_t = n_y y_t + (1 - n_y) \theta_y y_{t-1}$$

$$\hat{E}_t(y_{t+1}) = n_y E_t y_{t+1} + (1 - n_y) \theta_y^2 y_{t-1}$$

$$\hat{E}_t \pi_t = n_\pi \pi_t + (1 - n_\pi) \theta_\pi \pi_{t-1}$$

$$\hat{E}_t(\pi_{t+1}) = n_\pi E_t \pi_{t+1} + (1 - n_\pi) \theta_\pi^2 \pi_{t-1},$$

together with the alternative policy rule PR₂. Simplification yields the following reduced form model:

$$F x_t = B E_t x_{t+1} + C x_{t-1}, \quad (21)$$

for appropriate matrices F, B , and C , where $\det(B) \neq 0$ and $x = (y, \pi)'$. Equation (21) is the rational expectations model (the ARE) associated with the heterogeneous expectations model. Since the ARE has the same form as a rational expectations model with predetermined variables there is an established toolkit for determining determinacy, thereby making our approach easily accessible to practitioners.¹⁴ Techniques for analyzing the determinacy properties of a linear model are well-known: see for example Blanchard and Kahn (1980).¹⁵

Let

$$M = \begin{pmatrix} B^{-1}F & -B^{-1}C \\ I_2 & 0 \end{pmatrix}. \quad (22)$$

The determinacy properties of the model depend on the magnitude of the eigenvalues of M , which we write as λ_i , where $i > j \Rightarrow |\lambda_i| \leq |\lambda_j|$.¹⁶ Note that in general there are two predetermined variables, so that determinacy obtains when precisely two of the eigenvalues of M are outside the unit circle: fewer eigenvalues outside the unit circle implies indeterminacy and more implies explosiveness. As we will see in the sequel, the notion of indeterminacy can be refined in a useful way: following Evans and McGough (2005a), we say that if $|\lambda_1| < 1$ then the model is *order two indeterminate* (or, equivalently, *exhibits order two indeterminacy*) and if $|\lambda_1| > 1 > |\lambda_2|$ we say the

¹⁴As mentioned above, in rational expectations models, non-explosiveness is typically required in order to avoid violation of asymptotic first order conditions. Because a portion of our agents are rational and therefore have analogous asymptotic first order conditions, we also focus on non-explosive equilibria: see the Appendix.

¹⁵Because we have imposed perfect foresight for our rational agents, indeterminacy in this model will correspond to the presence of multiple bounded perfect foresight paths.

¹⁶Additive noise in the IS or AS relation does not impact the determinacy properties.

model is *order one indeterminate*. Note that in a general rational expectations model, the order of the indeterminacy corresponds to the largest possible dimension of a martingale difference sequence that can coordinate an associated sunspot equilibrium: for a detailed discussion see Evans and McGough (2005a). Finally, the same analysis applies in case policy rule two is used with the only modification being the dependence of the matrix M on the model's deep parameters.

Analysis of the determinacy properties of the associated rational model, given by (21), is straightforward and one can proceed by employing the usual solution techniques. However, the heterogeneity in our model raises an additional concern: how do we know that the aggregate time-series determined by (21) can also arise by aggregating from household plans? That is, a heterogeneous expectations equilibrium must simultaneously solve (21) and the individual agents' first order conditions. Given a solution to (21), it is always possible to back out a household plan for each individual since aggregate output is a weighted average of consumption across types. However, it also must be verified that, along an equilibrium path, bond holdings satisfy the transversality condition for each agent. See the Appendix for details concerning this verification.

3.4 Numerical Analysis of Determinacy

Numerical results require assigning values to the reduced form model's parameters, i.e. to $\lambda_1, \lambda_2, \gamma, \sigma$, and β , as well as the proportions of rational agents n_x and the values of the adaptation parameters θ_x . These parameters could be estimated, however, such an estimation is beyond the scope of this simple application. Instead, we "derive" values for our parameters using estimates available in the literature; and we test for robustness of our results to varying parameter values whenever possible.

The reduced form parameters of the usual New Keynesian model (i.e., the model under rational expectations) are given by λ (where $\lambda = \lambda_1 + \lambda_2$) and σ^{-1} . These parameters have been estimated by a number of authors including Woodford (1999), Clarida, Gali, and Gertler (2000) and McCallum and Nelson (1999):¹⁷ see Table 1. We use these authors' estimate of σ^{-1} , and set, as our benchmark, $\lambda_i = \lambda/2$.¹⁸ We test the robustness of this latter assumption by allowing the relative sizes of the λ_i to vary. There are also several estimates of γ available: we use as a benchmark $\gamma = .65$ as given by Walsh (2003), and again, test for robustness. The discount factor β is taken to be .99.

Given reduced form parameter values, the model is closed by specifying a policy rule (i.e. setting values for α_x and α_π), and choosing values for $\theta = (\theta_y, \theta_\pi)'$ and

¹⁷Calibrations are for quarterly data with i_t and π_t measured as quarterly rates. The CGG calibration (based on annualized rates) is adjusted accordingly

¹⁸It is straightforward to compute that $\lambda_i > 0$.

Table 1: Calibrations

Author(s)	σ^{-1}	λ
W	1/.157	.024
CGG	4	.075
MN	.164	.3

$n = (n_x, n_\pi)'$. To perform our analysis, we fix θ and then for $n_* \in \{1, .99, .9, .7\}$ we analyze the determinacy properties of models characterized by the following policy rules: $0 < \alpha_\pi < 2$, $0 < \alpha_x < 2$. We refer to this region as the benchmark policy space. Finally, in the sequel, we will say that certain numerical results “obtain robustly.” By this, we will mean that they obtain under the following conditions:

- For both policy rules;
- Across calibrations for benchmark γ and λ_i , and for specified θ and n ;
- Across $\lambda_2 \in \{.005, .012, .022\}$, for the W-calibration, benchmark γ , and for specified θ and n ;
- Across $\gamma \in \{.4, .65, .9\}$, for the W-calibration, benchmark λ_i , and for specified θ and n .

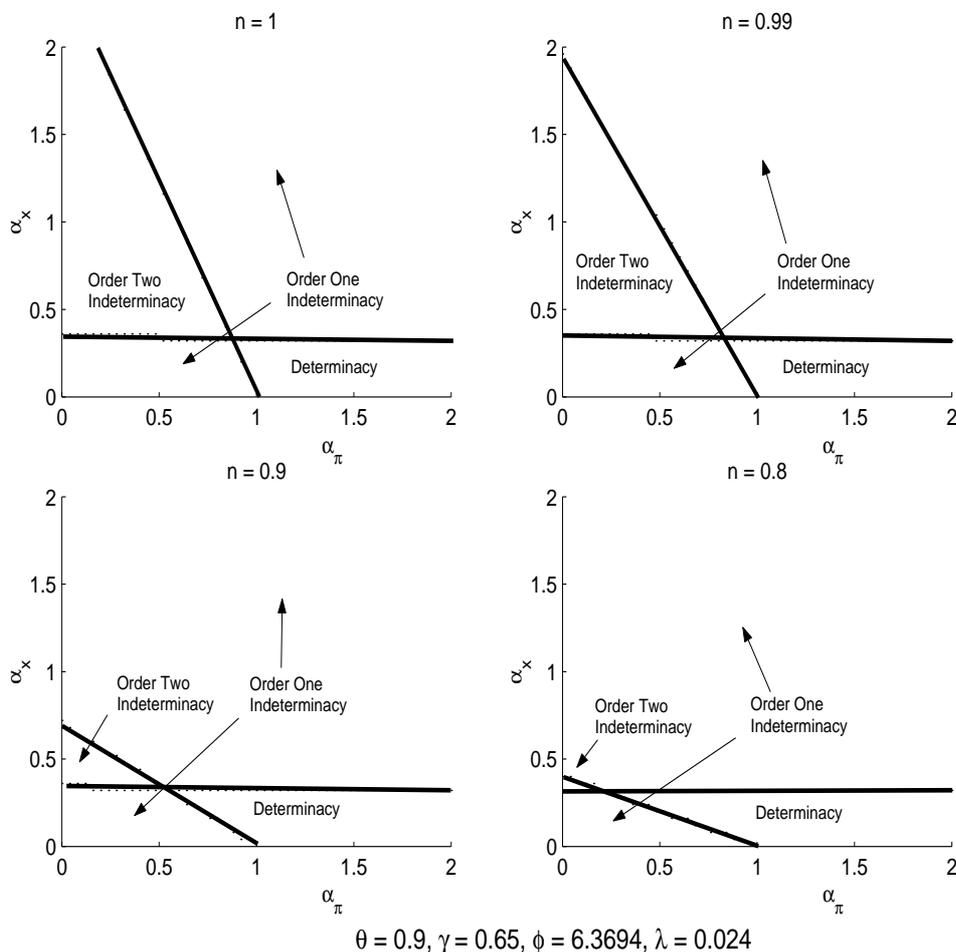
3.4.1 Adaptive Expectations: $\theta < 1$.

We first consider the case where adaptive agents place less weight on past data: that is, when $\theta < 1$. Figure 1 presents results for the Woodford calibration under the benchmark assumptions, with $\theta = .9$.¹⁹ The NW panel indicates the outcome under rationality, and is consistent with the results in Evans and McGough (2005). Much of the parameter space corresponds to indeterminacy of both order one and order two, with determinacy prevailing only for large α_π and low α_x . Now notice that as n decreases, the region corresponding to determinacy increases in size (as a proportion of the benchmark policy space). Thus we obtain the following result.

Result 3 *If $\theta < 1$ then policy rules corresponding to indeterminacy when $n = 1$ may yield determinacy when there is even a small proportion of adaptive agents in the economy. In this sense, the presence of adaptive agents may be stabilizing.*

¹⁹For all of our examples, $n_y = n_\pi$.

Figure 1: Determinacy Properties: $\theta < 1$.



As indicated in the NW panel, with full rationality, indeterminacy comprises a large portion of the parameter space. To avoid indeterminacy, the Taylor principle instructs policy-makers to “lean against the wind,” by setting the coefficient on inflation (or expected inflation) in the interest rate rule larger than one. In this way, a rise in inflation is met with a larger rise in the nominal rate, and through the Fisher relation this implies a rise in the real interest rate that will have a dampening effect on the economy.

While the Taylor principle does, in many cases, render a model determinate, and while it is wonderfully elegant and wholly intuitive, it is known in general to be neither necessary nor sufficient to guarantee a unique equilibrium: see Bullard and Mitra (2002), Benhabib and Eusepi (2005), Carlstrom and Fuerst (2005).²⁰

²⁰Benhabib, Schmitt-Grohe and Uribe (2003) show that even when it does yield determinacy, a rule satisfying the Taylor principle may be ill-advised.

Both the Taylor principle, and the fact that it may be neither necessary nor sufficient for determinacy is evident in Figure 1. The NW panel contains a steep, negatively sloped line anchored to the horizontal axis at $\alpha_\pi = 1$. For $\alpha_x = 0$, this line marks the boundary between rules that satisfy the Taylor principle and those that do not. This panel shows that rules which satisfy the Taylor principle, but which also set $\alpha_x > \approx .35$ result in order one indeterminacy: here the Taylor principle is not sufficient to prevent indeterminacy. It is also evident from the NW panel, that for values of α_π which are smaller than but very near to one, there correspond α_x so that the model is determinate, thus demonstrating that the Taylor principle is not necessary for determinacy to obtain.

Now notice the impact of heterogeneity: as n decreases the line anchored at $\alpha_\pi = 1$ rotates counterclockwise, thereby enlarging the relative size of the determinacy region; this rotation therefore increases the number of rules that simultaneously violate the Taylor principle and yield determinacy. We find that this *counterclockwise* rotation of the line anchored at $\alpha_\pi = 1$ obtains robustly.

3.4.2 Extrapolative Expectations: $\theta > 1$.

We now turn to the case where agents place greater weight on past data, i.e. $\theta > 1$. Here there is an additional subtlety: for non-explosive time paths of output gap and inflation to correspond to optimal behavior on the part of adaptive agents, it must be the case that $\hat{\theta}$, the parameter used by adaptive agents to forecast the current value of future bond holdings, be less than β^{-1} .²¹ To construct Figure 2, we take $\theta_\pi = \theta_y = \hat{\theta} = \beta^{-1} - \varepsilon$, for some $\varepsilon > 0$, and we consider a case in which $\theta_\pi \neq \theta_y = \hat{\theta}$ in Figure 3.²² Figure 2 presents results for the benchmark parameter values. The NW panel is identical to the panel in Figure 1, but here, as n decreases, the line anchored at $\alpha_\pi = 1$ rotates *clockwise*, thereby reducing the relative size of the region in parameter space corresponding to determinacy. However, for sufficiently high fractions of adaptive agents (small n) the presence of heterogeneity may be stabilizing, in accordance with Result 3. This leads us to the next result.

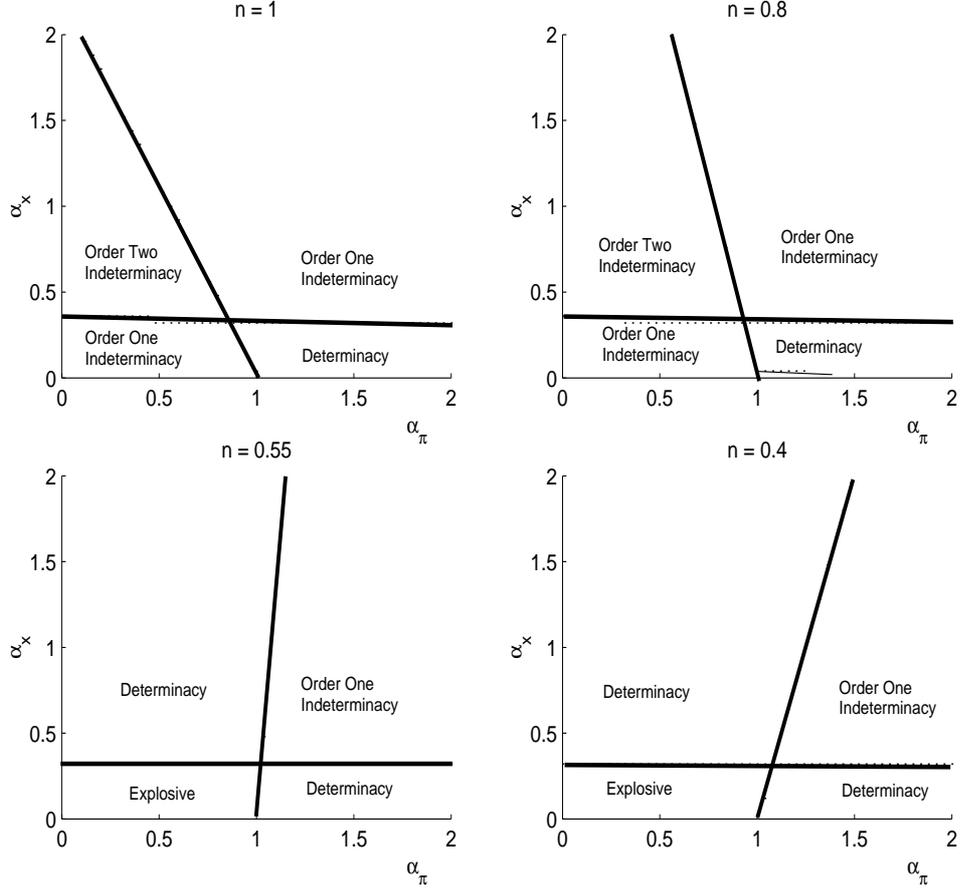
Result 4 *If $\theta > 1$ then policy rules that yield a determinate outcome in case of rationality may present indeterminacy in case even a very small proportion of agents are behaving adaptively. In this case, we find that the presence of adaptive agents may be destabilizing.*

²¹This ensures the satisfaction of the relevant transversality condition.

²²Since $\hat{\theta}$ is a forecast on the “household side” it seems natural to assume $\theta_y = \hat{\theta}$, and it is plausible to consider cases where $\theta_\pi \neq \theta_y$.

We find that this clockwise rotation line anchored at $\alpha_\pi = 1$ qualitatively obtains robustly.²³

Figure 2: Determinacy Properties: $\theta > 1$.



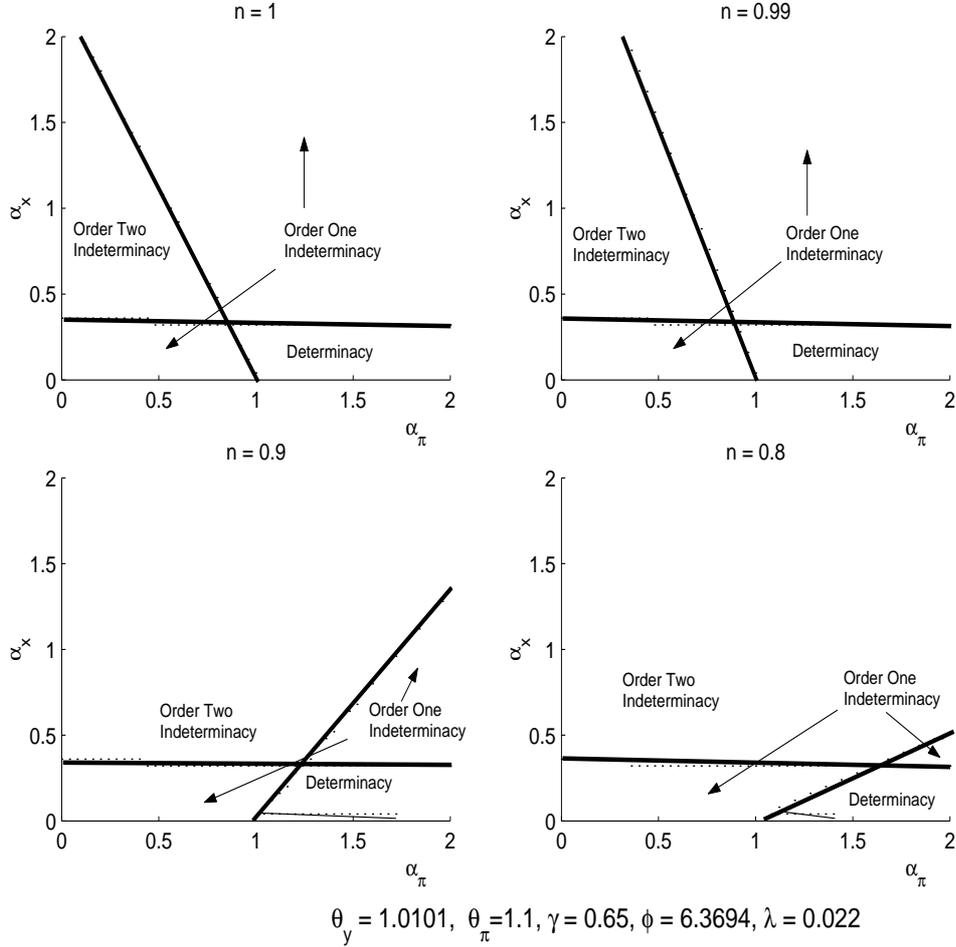
$$\theta = 1.0101, \gamma = 0.65, \phi = 6.3694, \lambda = 0.022$$

Whether heterogeneity is stabilizing or destabilizing, for a given fraction of rational agents, depends critically on the magnitude of the adaptation parameter θ . As an example, consider Figure 3, in which $\theta_\pi = 1.1$ and $\theta_y = \hat{\theta} = \beta^{-1} - \varepsilon$ as before. Here we are assuming that adaptive agents forecast π to grow at a faster rate than the present value of their bond holdings. We see that significant rotation obtains even if as few as 1% of private agents are adaptive, and if 10% are adaptive then indeterminacy obtains for some rules which satisfy the Talyor principle and yield

²³In the NE panel, there is a small unlabeled region along the horizontal axis and bounded above by a thin, downward-sloping line. In this region order two indeterminacy obtains.

determinacy under rationality.²⁴ Moreover, the clockwise rotation of the determinacy frontier expands the region of order two indeterminacy.

Figure 3: Determinacy Properties: $\theta > 1$.



Results 1 and 2 suggest that whether heterogeneity stabilizes or destabilizes depends on the distribution of agents across rational and adaptive expectations, and how strongly agents project past data in the adaptive predictor. When agents are adaptive in the more traditional sense, the region of determinacy may be more expansive. However, even slightly extrapolative or trend-chasing agents may destabilize a model that would be determinate under rationality. These results imply that a central bank uncertain about the precise nature of heterogeneity may desire policy robust for all reasonable forms of heterogeneity.

²⁴In the SW and SE panels, there are small unlabeled regions along the horizontal axis and bounded above by thin, downward-sloping lines. In these regions order two indeterminacy obtains.

The intuition for these results is clear. The usual ‘Taylor principle’ intuition is that if nominal interest rates are not adjusted more than one for one with expected inflation then aggregate supply shocks (or self-fulfilling shocks to inflation expectations) will be further propagated through lower real rates leading to higher contemporaneous and future inflation. In the heterogeneous expectations case the degree of this future propagation depends on n and θ (i.e., how strongly adaptive agents extrapolate past data). In the case of $\theta < 1$, these adaptive beliefs are mean reverting and will not further propagate the shock into higher future inflation. Thus, a more tepid response in the nominal interest rate rule is consistent with stabilization. When $\theta > 1$, though, adaptive agents are trend-following and so inflation will be placed on a self-fulfilling path unless policy is particularly vigilant against inflation. Notice that this logic is consistent with policymakers’ concerns that inflation expectations might become unhinged and out of the Fed’s control.²⁵

3.5 Discussion

The monetary policy literature, though wide and diverse, typically settles on the same recommendation: set policy so that the REE is determinate. At the heart of this recommendation is the property that a determinate steady state mitigates the potential for multiple equilibria to cause volatility of inflation and output. Our results indicate that a policy rule designed to implement determinacy may lead to inefficient outcomes if there exists heterogeneous expectations.²⁶

To illustrate this point in the starkest terms, we parameterized the model so that it is determinate under rational expectations. We showed that with even a very small fraction of agents the determinacy properties may be very different. If agents extrapolate past data, then policy set to ensure a determinate REE may lead to indeterminacy with inefficient inflation and output volatility. Therefore, the results of this paper demonstrate that if policy attempts to achieve a determinate REE in a New Keynesian model and these heterogeneous expectations dynamics are present, the policy-maker may unwittingly destabilize the economy. As such, these results are a cautionary note and suggest that perhaps policy should guard against indeterminacy in heterogeneous expectations models.

²⁵That values of $\theta < 1$ tend to be stabilizing and $\theta > 1$ tend to indeterminacy and instability suggests a model where there is parameter learning and dynamic predictor selection. In this setting it would be interesting to see if in a calibrated version of the model, whether θ_t would tend to a number above 1 pushing the economy into the indeterminacy region. This would then add heterogeneous expectations as a potential explanation of the Great Inflation and Moderation. We leave such an examination to future work.

²⁶It has also been emphasized that policy rules should be chosen so that the associated unique equilibrium is stable under learning: see for example, Bullard and Mitra (2002), Honkapohja and Mitra (2005) and Evans and McGough (2005). We note that the rule used to generate the plots in Figure 8 does produce an equilibrium that is stable under learning.

Whether heterogeneity is stabilizing or destabilizing depends on the distribution and nature of the heterogeneity. If adaptive agents' expectations are dampening then policy does not have to be quite as vigilant against inflation. Instead, when adaptive agents extrapolate or 'trend-chase' then policy needs to be even more aggressive in its stance against inflation. These results are intuitive and seem to align with policymaker concerns such as Bernanke (2004) who emphasizes that adaptive beliefs becoming untethered from policy changes poses a significant challenge. However, there are important open questions such as what specification of heterogeneous expectations is consistent with economic data. The benefit of our axiomatic approach is that it is sufficiently general that it nests many forms of heterogeneity besides rational versus adaptive. However, the motivating example presented in this Section demonstrates that heterogeneity can strongly impact a New Keynesian model.

4 Conclusion

This paper axiomatized a heterogeneous expectations version of a New Keynesian model from a micro-founded monetary economy with nominal rigidities. The heterogeneous expectations model aggregates into a reduced form whose primary distinction from the representative agent model is that conditional expectations are replaced by a convex combination of expectation operators. Depending upon the beliefs of boundedly rational agents, the New Keynesian aggregate supply relation may depend on expectations of current and future inflation and output. As an example illustrating the potential implications of our approach, this paper also examined the impact of expectations heterogeneity on a model's determinacy properties. Our central findings are two-fold. Heterogeneity may be stabilizing or destabilizing, depending on the nature of the adaptive expectations mechanism. In case the mechanism is extrapolative, models which are determinate in case of rationality may be indeterminate and hence exhibit sunspot equilibria in the presence of even a small proportion of adaptive agents.

The theory presented here aims to provide greater focus on the assumptions required for a New Keynesian model to incorporate boundedly rational behavior. Our primary theoretical result is to provide the assumptions on beliefs necessary to represent a heterogeneous agent economy with an aggregate law of motion. From an empirical viewpoint, our theoretical analysis is important as recent studies have documented heterogeneous expectations and 'disagreement' of beliefs in survey data (e.g. Carroll (2003), Mankiw, Reis, and Wolfers (2003), and Branch (2004)). However, how these divergent beliefs might impact reduced form relations governing the evolution of the economy has been an open issue. We illustrate that aggregation is possible provided expectations operators are linear, fix observables, satisfy the law of iterated expectations, and share common beliefs on differences in limiting wealth.

Appendix

The transversality condition requires that

$$\lim_k E_t^\tau \beta^k u_c(C_{t+k}^\tau) b_{t+k}^\tau \leq 0. \quad (23)$$

is satisfied, along an equilibrium path, at each t and for each agent type τ . We have not yet specified explicitly how type $\tau = 2$ “adaptive” agents forecast $u_c(C_{t+k}^2) b_{t+k}^2$. Since this is a forecast on the household side, and $\tau = 2$ are adaptive agents, it seems natural to assume they forecast this term via

$$E_t^2 u_c(C_t^2) b_t^2 = \hat{\theta} u_c(C_{t-1}^2) b_{t-1}^2,$$

for some value $\hat{\theta}$, and the forecast function is constructed iteratively as before.²⁷ *A priori*, there is no reason to expect $\hat{\theta} = \theta_y = \theta_\pi$. Below, we restrict attention to the case where $\hat{\theta} = \theta_y$. It follows that (23) is satisfied for adaptive agents provided that $\hat{\theta} < 1/\beta$.

We also have to ensure rational agents satisfy their transversality condition. To check this condition, first forward iterate the Euler equation for rational agents, to obtain

$$\hat{\Omega}_t^1 = -\sigma E_t \sum (i_{t+1} - \pi_{t+k+1}). \quad (24)$$

Since i_t, π_t are aggregate processes that, along with y_t , are nonexplosive solutions to (21), it follows that we may write $i_{t+1} - \pi_{t+k+1} = A z_{t+k}$, where z_t is a nonexplosive VAR in current and lagged output and inflation.²⁸ It follows from (24) that $\hat{\Omega}_t^1$ is uniformly bounded provided the aggregate variables are uniformly bounded as well. Next, we note that Ω_t^1 is non-explosive since it is smaller than Y_t , which itself is non-explosive. Then $\omega_t^1 = \log(\Omega_t^1/\bar{\Omega}^1)$ is non-explosive. Since $\hat{\Omega}_t^1$ is non-explosive, it follows that $\beta^{-1} b_{t-1}^1 - b_t^1 = \hat{\Omega}_t^1 - \omega_t^1$ is non-explosive, so that b_t^1 is expected to grow at most at a rate slower than β^{-1} . Thus, any process for bonds supporting a non-explosive equilibrium time series will satisfy the associated transversality conditions. We conclude, therefore, that given our specification for adaptive expectations, a nonexplosive solution to (21) is a heterogeneous expectations equilibrium.

²⁷Here agents are forecasting the utility foregone by saving by linearly extrapolating last period’s value.

²⁸The notion of “nonexplosiveness” used here can be made formal in a variety of ways depending on the type of extrinsic noise processes under consideration. For example, one can restrict attention to covariance stationary processes, or impose the weaker restriction of conditionally uniformly bounded processes. See Evans and McGough (2005c) for details.

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