Central-bank digital currency, deposit insurance and risk sharing in a monetary union

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Abstract

This paper studies the welfare implications of introducing a central-bank digital currency in a monetary union, in particular through its effects on resource allocation and risk sharing among the countries that form the monetary union. We construct a model where central-bank digital currency is risk-free and not remunerated, and competes with risky, interest-bearing, deposits. A key element of the model is that the banking sector may be integrated or fragmented across countries. We assess how the welfare effects of a central-bank digital currency depend on the banking integration (or fragmentation) among countries within the monetary union. We show that banking fragmentation may provide a rationale for the adoption of a central-bank digital currency (CBDC). Comparing the introduction of a CBDC with a federal deposit insurance scheme, we show that a CBDC is preferable if the economy is subject to significant macroeconomic shocks.

1 Introduction

Central-bank digital currencies, that are part of the agenda of most central banks around the world, are intended to help central banks move into the 21st century payments landscape. By now, central banks issue two forms of money: reserves, only available to banks, and physical cash, available to everyone, but increasingly inconvenient to perform transactions in a highly digitalized world. By issuing such digital currencies, central banks

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would indeed offer a payment instrument aligned with current technological and societal developments.

The policy debate on the convenience and usefulness of central-bank digital currencies (CBDCs) has been intense over the last years, as policy concerns are as widespread as the beliefs on the benefits for society that these currencies could bring. On the positive side, CBDCs could allow central banks to ensure monetary sovereignty vis-a-vis private alternatives such as cryptocurrencies, limit banks' market power by providing citizens with a safe digital means of payment and foster financial inclusion. On the negative side, CBDCs could bring bank disintermediation, thereby shrinking bank lending to the real economy, and even promote financial instability by facilitating the occurrence of bank runs.

In this paper, we seek to study the welfare effect of a central-bank digital currency in a monetary union. We believe that that effect depends to a large extent on the characteristics of the banking sector, and in particular its integration (or fragmentation) among the countries that form the union. Looking at major economies, there is a stark contrast between the determination expressed by the European Central Bank to issue a CBDC in the euro area and the lack of political enthusiasm regarding CBDCs in the United States. Interestingly, both economies are monetary unions where states share a central bank and a common currency. However, while in the U.S. banking markets are integrated among states, this is not the case in the euro area where hurdles to bank integration across the borders persist years after the inception of the euro. Due partly to national regulatory barriers to cross-border bank capital and liquidity reallocation, there is a strong home bias in the European banking sector (Gotti *et al* 2024).[1] Importantly, while the U.S. has a federal deposit insurance, in the euro area the enactment of a deposit insurance at the federal level is a pending matter. In the policy debate, the usual argument in favor of the digital euro is not about banking integration, but relates to sovereignty concerns regarding payments ecosystems.[2] In this paper, we highlight the lack of financial integration within a monetary union as another, overlooked, rationale for the issuance of a CBDC, as we explain below.

To study the introduction of a CBDC within a monetary union, we build a model based on Lagos-Wright (2005) and Rocheteau-Wright (2005), with banks as in Berentsen, Camera and Waller (2007). There are two countries populated by buyers and sellers, who trade a consumption good, and entrepreneurs and suppliers, who trade capital. Banks grant loans to entrepreneurs and issue deposits. Countries are *ex ante* identical, although

[1] In 2024, cross-border (intra euro area) retail loans represent approximately 8% of all retail loans, and the fraction of cross-border deposits is approximately 7%.

[2] For instance, P. Lane, member of the ECB's Executive Board, has declared in a speech on the digital euro: "Europe's reliance on foreign payment providers has reached striking levels. International card schemes such as Visa and Mastercard now process 65% of euro area card payments. In 13 out of 20 euro area countries, national card schemes have been entirely replaced by these international alternatives." (P. Lane, ECB, March 2025). The sovereignty concerns have several dimensions, like geopolitical risks, implications for monetary policy, costs for merchants and businesses and data monetisation.

in a given period one country becomes riskier than the other: being a "risky" country means that there is a positive probability of a negative macroeconomic shock that reduces output produced by entrepreneurs, in that period in that country. In that case, a fraction of entrepreneurs fail to repay their loans. To consume, buyers need a "digital" means of payment: bank deposits or, if available, a central-bank digital currency. Buyers are risk averse, and so they have a motive for smoothing consumption across the different states of the economy. Bank deposits, if they are partly backed by loans taken out by entrepreneurs in the risky country, are risky. Central-bank money, shared by the two countries, is risk-free. When a digital form of central bank money is available, the key choice for buyers is about how much to hold of interest-bearing, potentially risky, deposits, and how much of unremunerated, risk-free, central-bank digital currency.

Two settings regarding the banking sector are considered: one with banking markets that are integrated across the borders, and hence admits cross-border operations (loans and deposits), and one with fragmented banking markets where agents can only contract with domestic banks (all other markets are perfectly integrated across countries). In both cases, banks are perfectly competitive. In equilibrium, with an integrated banking sector, all banks grant loans to both the risky entrepreneurs (located in the country that turns out to be risky in a given period) and safe entrepreneurs (located in the country with no macroeconomic risk). If the negative macroeconomic shock occurs, banks experience losses that are ultimately borne by depositors in both countries. Irrespective of their risky or safe profile, entrepreneurs are able to invest the same amount of capital in the two countries. The reason is that banks only care about the good state of the economy; as they do not internalize the effects of loans default they have no incentives to reduce lending to risky entrepreneurs.

With fragmented banking markets, the equilibrium changes substantially. Banks in the safe country only lend to domestic, safe, entrepreneurs, and banks in the risky country only lend to risky entrepreneurs. As a result, deposits are risk-free in the safe country, but risky in the risky country. Deposits in the risky country are even riskier than with integrated banking, since the asset portfolio of risky banks is now less diversified.

Our results are as follows. First, we compute the maximum level of risk (i.e., the probability of a negative macroeconomic shock in the risky country) that leads depositors to strictly prefer deposits to the central-bank digital currency. Above such level of risk, depositors would instead hold a portfolio of deposits and CBDC to partially insure against the risk of bank failure. We show that that maximum level of risk is smaller with fragmented banking than with integrated banking. Intuitively, losses borne by depositors are greater with fragmented banking if the negative macroeconomic shock realizes, and therefore agents have a stronger motive to hold CBDC. Then, we assume that initially no CBDC is available and compare the effects of the introduction of a CBDC for both considered configurations of the banking sector. We show that, over a range of risk levels, introducing a CBDC has no welfare effect with integrated banking markets since agents still strictly prefer holding interest-bearing, and not too risky, deposits. By contrast, introducing a CBDC unambiguously improves social welfare when banking markets are fragmented, since it allows agents in the risky country to smooth consumption more efficiently across the states of the economy.

To complement our analysis, we compare the welfare level attained by introducing a central-bank digital currency with the one obtained by implementing a federal deposit insurance in a monetary union. This comparison is relevant because these two supranational policies share the objective of rendering the means of payment more uniform across countries within the monetary union. The central-bank digital currency achieves this by providing a common central-bank issued form of money for agents in all member states, while a federal deposit insurance tends to equalize the level of risk of deposits issued in different countries. A federal deposit insurance has the potential of achieving higher consumption risk sharing among depositors located in different countries by effectively insuring deposits against losses. From this point of view, a federal deposit insurance is superior to a CBDC. However, it does not lead to a reduction in inefficient lending to risky firms as CBDC does. The relative weights of these two effects vary with the level of output losses that result from macroeconomic shocks. We show that, for small macroeconomic shocks welfare is maximized with a federal deposit insurance scheme rather than by introducing a CBDC, while the opposite is true for large macroeconomic shocks.

Related literature. A growing literature studies the effects of a retail CBDC on the banking sector.[3] Several papers like Andolfatto (2021) and Chiu *et al* (2023) study the introduction of a CBDC that competes with deposits issued by imperfectly competitive banks. The introduction of CBDC acts as an outside option for depositors and effectively reduces the market power of banks that respond by increasing deposit rates to retain depositors. Thus, while CBDC reduces banks' profits, bank disintermediation need not arise since the increase in deposit rates may lead to a higher volume of deposits. Keister and Sanches (2023) consider different designs of CBDC, including the one in which CBDC is a competitor to bank deposits and bears interest. CBDC may therefore lead to bank disintermediation, but its remuneration leads to higher efficiency in exchange and through this effect it improves welfare. Brunnermeier and Niepelt (2019) study CBDC as an application of the coexistence between public and private money. They consider explicitly the balance sheet of the central bank to determine the conditions for an allocation equivalence when a CBDC is introduced. In most papers that study the coexistence between CBDC and banks deposits, both means of payment are risk-free. We contribute to this literature by explicitly considering the competition between risky deposits, as they are issued by banks subject to a risk of failure, and central-bank digital money that is inherently risk-free. Unlike, among others, Andolfatto and Chiu *et al* who consider an interest-bearing CBDC, we model an unremunerated CBDC. Williamson (2022) builds a model that accounts for the risk of banks' failure, in order to assess how the introduction

[3]For recent reviews of the literature, see Ahnert *et al* (2024a) or Bindseil and Senner (2024).

of a CBDC impacts the occurrence of bank runs. Ahnert *et al* (2024b) analyze the relation between financial instability and CBDC remuneration, while Monnet *et al* (2025) study how a remunerated CBDC affects banks' risk taking. By studying the issuance of a CBDC in a monetary union, and highlighting the role of the integration of banking markets across the countries that form the union, we provide a different angle of analysis from those already considered in the literature.

The rest of the paper is organized as follows. Section 2 presents the model. Sections 3 and 4 describe the equilibrium with integrated banking markets and with fragmented banking markets, respectively. In Section 5 we study the welfare implications of introducing a CBDC in both settings. Section 7 concludes. All proofs are relegated to the Appendix.

2 Environment

The environment builds in the sequential-market model developed by Lagos and Wright (2005) and Rocheteau and Wright (2005). Time is discrete and lasts forever. Every period is divided into three subperiods. In the first subperiod suppliers and entrepreneurs trade in a *capital market*, while in the second buyers and sellers trade in a *goods market*. In the third subperiod a numeraire good is traded in a *settlement market*. Time across periods is discounted with factor $\beta \in (0,1)$. The economy is composed of two identical countries. There are four types of agents in every country (each with unit measure): infinitely-lived suppliers, buyers and sellers and one-period lived entrepreneurs.

In the goods market, buyers get utility $u(q)$ from consuming q goods, where $u'(q) > 0$, $u''(q) < 0$, $u'(0) = \infty$ and $u'(\infty) = 0$. For tractability, we set $u(q) = \ln(q)$. Sellers can produce q at linear cost. In the settlement market, buyers and sellers can produce and consume with linear utility (a negative consumption quantity is interpreted as production).

Suppliers can produce capital k with linear cost to sell in the capital market. Capital is storable within a period but not across periods. Entrepreneurs have access to a technology to invest capital in the first subperiod and obtain numeraire goods in the third subperiod as explained below. Entrepreneurs are born with no endowment. Suppliers and entrepreneurs obtain linear utility from consumption in the settlement market.

At the beginning of every period, the first country is "safe" and the second country is "risky" (with probability 1/2), or the first country country is "risky" and the second country is "safe" (with probability 1/2). In the safe country, entrepreneurs who invest k at the beginning of the period produce $f(k)$ units of the numeraire in the settlement market. The production function $f(k)$ is concave with $f'(k) > 0$, $f''(k) < 0$, $f'(0) = \infty$ and $f'(\infty) = 0$. For simplicity, we generally assume that $f(k)$ takes the form $f(k) = Ak^a$ with $0 < a < 1$.

Being a risky country means that there is a probability θ_0 that a bad macroeconomic shock hits the economy. Conditional on the realization of the bad macroeconomic shock, only a fraction $(1-\theta_1)$ of entrepreneurs who have invested k produce $f(k)$ in the settlement market, while a fraction θ_1 of entrepreneurs produce zero output. With probability $(1-\theta_0)$, the bad macroeconomic shock does not realize and all entrepreneurs produce output. Let $\theta = \theta_0\theta_1$.

In every country there are competitive banks, similarly as in Berentsen, Camera and Waller (2007). Banks issue deposits at the beginning of the period and commit to redeem these deposits at the end of the period, during the settlement market. Deposits can be used as means of payment in the goods market and the capital market. To motivate the use of fiat money we assume however that suppliers are unbanked which implies that, to back their deposits, banks must keep reserves (central-bank money, either physical or digital). Banks can grant loans to entrepreneurs at the beginning of the period and can seize output in the settlement market to enforce loan repayment.[4] Banks' assets (loans) are perfectly observable.

There is a common central bank that issues fiat money at a money growth rate $\gamma = M/M_{-1}$, where M is the money stock in t (and M_{-1} is the money stock in $t-1$). Newly-issued money is used to make lump-sum transfers in the settlement market. We consider two alternative scenarios regarding money issuance: the central bank may issue physical currency or a central-bank digital currency (CBDC).

To motivate the introduction of a CBDC, we assume that transactions in the goods market require an electronic means of payment (e.g., because the seller and the buyer trade remotely). Therefore, buyers may use bank deposits or the central-bank digital currency as means of payment if the latter is available.[5]

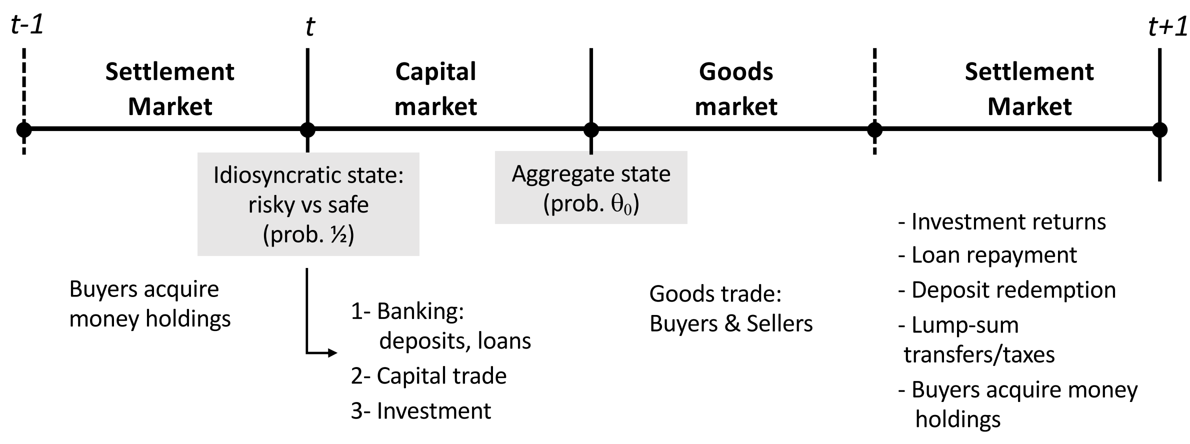

Figure 1: Timing of the model

A key aspect of the paper is the *integration* or, alternatively, the *fragmentation* of

[4]In the Appendix we present an extension of the model with money creation where only a fraction of suppliers require the use of central-bank money and thus deposits may be issued through lending.

[5]The assumption that all transactions in the goods market require an electronic means of payment is made for simplicity (making it for a fraction of transactions would suffice for our analysis).

the banking markets across countries. In the model the banking sector performs two activities, deposits and loans. With integrated banking markets, agents can contact all banks regardless of their country of residence: entrepreneurs are able to take out cross-border loans and buyers are allowed to acquire deposits issued by foreign banks. Under fragmented banking markets, instead, cross-border bank operations are not feasible and thus agents (buyers and entrepreneurs) can only contract with banks in their country. Notice that the capital and the goods markets are perfectly integrated irrespective of the fragmentation featured by the banking sector.

The timing of the model for a representative period t is summarized in Figure 1. The state of the each country (safe or risky) is revealed to all agents at the beginning of the period. Then buyers deposit money holdings with banks and entrepreneurs take out bank loans to purchase capital in the capital market. Upon the realization of the aggregate state that determines whether a macroeconomic shock hits the risky country or not, buyers and sellers trade in the goods market. Finally, during the settlement market banks redeem deposits, entrepreneurs sell part of their production to repay bank loans and consume the remaining goods, and buyers acquire fiat money to bring to $t+1$.

2.1 Efficient allocation

In order to have a benchmark for capital allocation and welfare, consider a social planner who has control over both countries and assigns equal weight to all agents. Consumption by buyers in the safe and risky country is respectively denoted by q and q^θ. Production by sellers in the safe (risky) country is denoted by q_s (q_s^θ). k_s (k_s^θ) is the quantity of capital produced by suppliers in the safe (risky) country and k (k^θ) is total capital invested in the safe (risky) country. Social welfare is

$$(1-\beta)\mathcal{W} = u\left(q\right) + u\left(q^\theta\right) - q_s - q_s^\theta - k_s - k_s^\theta + f\left(k\right) + (1-\theta)\, f\left(k^\theta\right)$$

since with probability θ_0 a fraction θ_1 of firms in the risky country produce zero output and $\theta_0\theta_1 = \theta$. The planner's choice is subject to the resource constraints

$$\begin{aligned} q_s + q_s^\theta &\geq q + q^\theta \\ k_s + k_s^\theta &\geq k + k^\theta \end{aligned}$$

The optimal allocation satisfies $u'\left(q\right) = u'\left(q^\theta\right) = 1$, with $q^\theta + q = q_s + q_s^\theta$, and $k_s + k_s^\theta = k + k^\theta$. The following Lemma defines the efficient capital allocation.

Lemma 1 *The optimal capital allocation solves*

$$f'(k) = 1 \tag{1}$$

$$(1-\theta)f'(k^\theta) = 1 \tag{2}$$

From Lemma 1, if $\theta > 0$ the optimal allocation satisfies $k > k^\theta$ since in this case expected productivity is higher in the safe country.

3 Integrated banking

In this section we consider a setting where banking markets are perfectly integrated across countries, implying that both cross-border loans and cross-border deposits are feasible. We first describe agents' decisions in the settlement market, at the end of a representative period, and then present the banks' problem at the beginning of every period. We use the equilibrium conditions for banks to describe the decisions of entrepreneurs and suppliers in the capital market and those of buyers and sellers in the goods market.

3.1 Settlement market

In the settlement market, all agents may produce and consume the numeraire good and rebalance money holdings. Banks redeem deposits and recover loan repayment from entrepreneurs.

With an integrated banking sector, banks are able to contract with agents in both countries, and hence banks' decisions are identical regardless of their location. Therefore there is only one type of deposit that we denote as d. Money issued by the central bank is simply generically referred to as "money", since it could be physical or digital, as it will be specified later. $W(m,d)$ denotes the value of entering the settlement market with m money balances and d deposits, for an agent in a given country. Let ϕ and ϕ^θ denote, respectively, the price of money in terms of the numeraire if the bad macroeconomic shock does not realize, with probability $(1-\theta_0)$, and if the shock occurs, with probability θ_0. $V(m)$ denotes the value at the beginning of a period for a generic agent who holds m units of money and x denotes consumption of the numeraire good.

The representative agent solves

$$W(m,d) = \max_{x,m_{+1}} x + \beta V_{+1}(m_{+1})$$

subject to the budget constraint which is, if banks do not fail,

$$x + \phi m_{+1} = \phi(1+i_d)d + \phi m + \phi T$$

where i_d is the interest rate on deposits and T is the lump-sum monetary transfer from the central bank. If case of failure, the budget constraint is instead

$$x + \phi^\theta m_{+1} = \phi^\theta \delta d + \phi^\theta m + \phi^\theta T$$

with δ the bank's liquidation value per unit of deposit that is defined later.

By inserting the budget constraints in the objective function, we obtain the first-order conditions on m_{+1}, that we can express one period lagged, as

$$\beta V_m = \phi_{-1}$$

in the absence of the macroeconomic shock, and as

$$\beta V_m = \phi^{\theta}_{-1}$$

if the shock occurs. We deduce that the price of money in terms of the numeraire is the same independently of the current realization of the macroeconomic shock; i.e., $\phi^{\theta}_{-1} = \phi_{-1}$ and $\phi^{\theta} = \phi$. This implies that money demand is also the same regardless of the current realization of the shock, and is also the same in the two countries.

The envelope condition on m gives

$$W_m = \phi \tag{3}$$

The envelope condition on d gives

$$W_d = (1-\theta_0)\,\phi\,(1+i_d) + \theta_0 \phi \delta \tag{4}$$

We focus on stationary monetary equilibria where real variables such as end-of-period real money holdings are time-invariant, which implies that

$$\gamma = \frac{\phi_{-1}}{\phi} \tag{5}$$

where the price of money in terms of numeraire evolves over time according to the money growth rate γ.

3.2 Banking activities

Banks are perfectly competitive—they take the loan interest rate i and the deposit interest rate i_d as given—and protected by limited liability. With integrated banking, banks thus choose the level of deposits d per depositor and the amounts of the individual loans ℓ and ℓ^{θ} granted to safe and risky entrepreneurs, respectively, to maximize profits, as follows

$$\begin{aligned} \max_{d,\ell,\ell^{\theta}} \quad & (1-\theta_0)\left[(1+i)\,\ell + \left(1+i^{\theta}\right)\ell^{\theta} - (1+i_d)\,2d + 2d - \left(\ell+\ell^{\theta}\right)\right] \\ & + \theta_0 \langle (1+i)\,\ell + (1-\theta_1)\left(1+i^{\theta}\right)\ell^{\theta} - (1+i_d)\,2d + 2d - \left(\ell+\ell^{\theta}\right)\rangle^{+} \end{aligned}$$

subject to $\ell + \ell^\theta \leq 2d$, with $\langle . \rangle^+$ our notation for the positive part of a real-valued expression.[6] With probability $(1-\theta_0)$, all entrepreneurs produce output and repay their loans, while with probability θ_0 all safe entrepreneurs and a fraction $(1-\theta_1)$ of risky entrepreneurs repay loans. Profits result from loan repayment by safe and risky entrepreneurs and reserves held in excess of withdrawals, net of repayment to depositors. With perfectly competitive banks protected by limited liability, in equilibrium banks' profits must be zero in the good state. This implies that if the bad state occurs banks necessarily make losses that are borne by depositors. The zero-profit condition in the good state and the reserve constraint, which binds in equilibrium, yield

$$(1+i)\,\ell + \left(1+i^\theta\right)\ell^\theta = (1+i_d)\left(\ell + \ell^\theta\right)$$

It follows that $i = i_d$ and $i^\theta = i_d$ and therefore

$$i = i^\theta = i_d \tag{6}$$

If the bad state occurs, the bank's liquidation value per unit of deposit δ satisfies

$$\delta = \frac{(1+i)\,\ell + (1-\theta_1)\left(1+i^\theta\right)\ell^\theta}{2d}$$

since all safe entrepreneurs and a fraction $(1-\theta_1)$ of risky entrepreneurs repay their loans. Since from (6) the loan interest rate is the same for all entrepreneurs, they all face the same problem implying that $\ell = \ell^\theta$. Using the resource constraint, δ becomes

$$\delta = (1-\theta_1/2)\,(1+i) \tag{7}$$

Since $\ell = \ell^\theta$ and $i = i^\theta$, it follows that $k = k^\theta$.

3.3 Capital market

The capital market is perfectly competitive and integrated across countries. To have the central-bank money that they need to buy capital from suppliers, entrepreneurs take out bank loans. We denote as k_s the amount of capital exchanged for money at a price p_k. Using (3), the suppliers' program for a given period can be written as[7]

$$\max_{k_s} -k_s + \phi p_k k_s$$

[6]That is, $\langle x \rangle^+ \equiv \max(x, 0)$. Note that there is a unit measure of depositors (buyers) in each country, and hence the total measure of depositors is 2.

[7]Note that buyers are the only agents who are willing to hold money across periods, as they have a concave utility function for consumption acquired in the goods market. Therefore we can abstract from choices across periods for suppliers and sellers.

Suppliers' indifference condition regarding their capital production k_s is

$$\phi p_k = 1 \tag{8}$$

In both countries, entrepreneurs solve

$$\max_{\ell,k} f(k) - (1+i)\phi\ell$$

subject to $\ell \geq p_k k$. Since this constraint holds with equality for $i > 0$, the first-order condition on k yields

$$f'(k) = \phi p_k (1+i) \tag{9}$$

Using (8), (9) becomes

$$f'(k) = 1 + i \tag{10}$$

where entrepreneurs' capital investment is determined by the interest rate.

3.4 Goods market

3.4.1 With CBDC

Suppose that CBDC is available. With CBDC, m can be used to buy goods in the market at a price p. Notice that sellers sell the good when all uncertainty is resolved. q_s^d and q_s^δ denote, respectively the amount of goods sold in exchange for deposits in the absence of a macroeconomic shock, in which case the price of goods is p_d, and in case the bad shock occurs, at a price p_d^δ. q_s^m denotes the amount of goods sold in exchange for money (similarly, q_m and q_d denote consumption bought with money and deposits, respectively, by buyers). In the absence of a macroeconomic shock, with probability $(1-\theta_0)$, sellers solve

$$\max_{q_s^d, q_s^m} -q_s^d - q_s^m + \phi p_d \left(1 + i_d\right) q_s^d + \phi p q_s^m$$

and, in case of a shock, with probability θ_0, sellers solve

$$\max_{q_s^\delta, q_s^m} -q_s^\delta - q_s^m + \phi p_d^\delta \delta q_s^\delta + \phi p q_s^m$$

with δ the bank's liquidation value per unit of deposit in case of failure. Therefore sellers' indifference conditions give

$$\phi p_d (1 + i_d) = 1, \tag{11}$$

$$\phi p_d^{\delta} \delta = 1, \tag{12}$$

$$\phi p = 1 \tag{13}$$

At the beginning of t buyers choose their consumption q, with $q = q_m + q_d$, and the amount of their deposits d, as follows

$$\max_{d, q_m, q_d, q^{\delta}} (1 - \theta_0) [u (q) + W (m - d - p q_m, d - p_d q_d)] + \theta_0 \left[u \left(q^{\delta}\right) + W^{\theta} (0, 0)\right]$$

subject to $d \leq m$, $pq_m \leq m - d$, $p_d q_d \leq d$ and

$$q^{\delta} = \frac{m - d}{p} + \frac{d}{p_d^{\delta}}$$

Let λ, μ_m and μ_d denote the Lagrange multipliers on the first, second and third constraints. Anticipating that the fourth constraint binds in equilibrium and using the envelope conditions (3) and (4) and the sellers' indifference conditions (11)-(13), the buyer's first-order conditions on q_m and q_d are

$$(1 - \theta_0) [u' (q) - 1] = \mu_m, \tag{14}$$

$$(1 - \theta_0) [u' (q) - 1] = \mu_d, \tag{15}$$

Therefore $\mu_d = \mu_m > 0$ for $u'(q) > 1$ and $\mu_d = \mu_m = 0$ for $u'(q) = 1$. Using also (6), (7) and (15), we obtain the buyer's first-order condition on d,

$$(1 - \theta_0) u' (q) i + \theta_0 u' \left(q^{\delta}\right) [(1 - \theta_1/2)(1 + i) - 1] = \frac{\lambda}{\phi} \tag{16}$$

Using (5), (13), (14) and (16), the first-order condition on money holdings for buyers is

$$\frac{\gamma}{\beta} = (1 - \theta_0) u' (q) (1 + i) + \theta_0 u' \left(q^{\delta}\right) (1 - \theta_1/2)(1 + i) \tag{17}$$

If $\lambda > 0$, buyers deposit their entire money holdings with banks, and we have $d = p_d^{\delta} q^{\delta}$. Using the banks' resource constraint, the sellers' indifference condition (12), the entrepreneurs' constraint and the entrepreneurs' first-order condition (9), this yields

$$q^{\delta} = \frac{k f' (k)}{1 + i} \delta$$

Using (7), it becomes

$$q^{\delta} \;\; = \;\; kf'(k)(1-\theta_1/2) \tag{18}$$

If $\mu_d > 0$, we further have that $d = p_d q$. Using the banks' resource constraint, the sellers' indifference condition (11), the zero-profit condition (6), the entrepreneurs' constraint and the entrepreneurs' first-order condition (9), we obtain

$$q = kf'(k) \tag{19}$$

If $\lambda = 0$, the buyer's deposit constraint is not binding and from (16) we obtain

$$(1-\theta_0)\,u'(q)\,i \;\; = \;\; \theta_0 u'\left(q^{\delta}\right)\left[1-(1-\theta_1/2)(1+i)\right] \tag{20}$$

which combined with (17) implies

$$\frac{\gamma}{\beta} \;\; = \;\; (1-\theta_0)\,u'(q) + \theta_0 u'\left(q^{\delta}\right) \tag{21}$$

If $\mu_m = \mu_d > 0$, the buyer's second and third constraint bind. Combining them with the buyer's fourth constraint, we obtain

$$q = q^{\delta} + d\left(\frac{1}{p_d} - \frac{1}{p_d^{\delta}}\right)$$

Using the banks' constraint[8], the entrepreneurs' constraint, the zero-profit condition in (6), the sellers' indifference conditions (11) and (12), the entrepreneurs' first-order condition (9) and the value of δ in (7), we obtain

$$q^{\delta} = q - kf'(k)\,\theta_1/2 \tag{22}$$

Using the assumed utility function $u = \ln$, the first-order condition on money holdings (17) becomes

$$\frac{\gamma}{\beta} = (1-\theta_0)\frac{1}{q}(1+i) + \theta_0\frac{1}{q^{\delta}}(1-\theta_1/2)(1+i) \tag{23}$$

and the indifference condition (20) becomes

$$(1-\theta_0)\,q^{\delta} i \;\; = \;\; \theta_0 q\left[1-(1-\theta_1/2)(1+i)\right]$$

In the equilibrium where $\lambda > 0$, using (10), (18) and (19), (23) yields

$$k = \frac{\beta}{\gamma} \tag{24}$$

[8] $\ell = d$ or, alternatively, $\ell + \ell^{\theta} = 2\ell = 2d$.

If $u'(q) = 1$, (18) combined with (23) gives

$$\frac{k}{(1-\theta_0)\,kf'(k)+\theta_0} = \frac{\beta}{\gamma} \tag{25}$$

In the equilibrium with $\lambda = 0$, with $u = \ln$, using (10), (20), (22) and (23) we obtain

$$q = \frac{\beta}{\gamma}\frac{f'(k)\,(1-\theta_0)\,\theta_1/2}{1-f'(k)\,(1-\theta_1/2)} \tag{26}$$

and

$$\frac{\gamma}{\beta}k\left[f'(k)-1\right] = \frac{f'(k)\,(1-\theta/2)-1}{1-f'(k)\,(1-\theta_1/2)} \tag{27}$$

that determines the value of invested capital k when buyers keep part of their money balances within the period.

If $u'(q) = 1$, (21) yields

$$q^\delta = \frac{\theta_0}{\frac{\gamma}{\beta}-1+\theta_0} \tag{28}$$

that we combine with (10) to obtain

$$\frac{\gamma}{\beta} = \frac{(1-\theta_0)\,f'(k)\theta_1/2}{1-f'(k)\,(1-\theta_1/2)} \tag{29}$$

3.4.2 Without CBDC

Without CBDC and with integrated banking, buyers deposit all their money holdings, since in t they have no more information than they had in $t-1$, money is costly to hold and physical cash is not suitable as a means of payment in the goods market. As in the case with CBDC, we anticipate that if the bad macroeconomic shock occurs buyers spend all their deposits to afford consumption.

The buyer's program is

$$\max_{d,q,q^\delta} \quad (1-\theta_0)\left[u(q)+W(m-d-pq_m, d-p_dq_d)\right]+\theta_0\left[u\left(q^\delta\right)+W^\theta(0,0)\right]$$

subject to $d \leq m$, $p_dq \leq d$ and $p_d^\delta q^\delta = d$.

Using (6), (7) and (11), the buyer's first-order conditions on q and d yield

$$(1-\theta_0)\left[u'(q)-1\right] = \mu_d$$

and

$$\frac{\mu_d}{p_d}+(1-\theta_0)\,\phi i_d+\theta_0\left[u'\left(q^\delta\right)\frac{1}{p_d^\delta}-\phi\right] = \lambda$$

where λ and μ_d are the Lagrange multipliers on the first and second constraints. Combining both equations we obtain

$$\left[(1-\theta_0)\,u'(q)+\theta_0 u'\left(q^{\delta}\right)(1-\theta_1/2)\right](1+i)-1 \;=\; \frac{\lambda}{\phi} \tag{30}$$

The first-order condition on money holdings is

$$\frac{\gamma}{\beta}=\frac{\lambda}{\phi}+1 \tag{31}$$

Using (30), it becomes

$$\frac{\gamma}{\beta}=\left[(1-\theta_0)\,u'(q)+\theta_0 u'\left(q^{\delta}\right)(1-\theta_1/2)\right](1+i)$$

From (31), $\lambda>0$ always holds in equilibrium if $\gamma>\beta$. The equilibrium is identical to the one in with CBDC for the case with $\lambda>0$ and $d=m$. This means that if initially there is no CBDC and then the CBDC is introduced, the economy may switch from the equilibrium with $\lambda>0$ to the equilibrium with $\lambda=0$. While with no CBDC the equilibrium with $\lambda>0$ exists for all parameter values, it may fail to exist if the CBDC is available.

3.5 Equilibrium with CBDC

We can now state the definition of the equilibrium under integrated banking with $\lambda>0$, in which the deposit constraint binds and depositors strictly prefer to hold deposits within the period rather than the unremunerated central-bank digital currency. For this, we replicate equations (10), (18), (19) and (24).

Definition 1 *An equilibrium with $\lambda>0$ is $\left\{k,i,q,q^{\delta}\right\}$ that solve, recursively,*

$$\begin{aligned}\frac{\beta}{\gamma} &= k\\ i &= f'(k)-1\\ q &= f'(k)\,k\\ q^{\delta} &= f'(k)\,k\,(1-\theta_1/2)\end{aligned}$$

If $u'(q)=1$, the equilibrium with $\lambda>0$ is $\left\{q^{\delta},k,i\right\}$ that solve (10), (18) and (25).

To state the definition of the equilibrium with $\lambda=0$ under integrated banking, we replicate the equations (10), (22), (26) and (27).

Definition 2 *An equilibrium with* $\lambda = 0$ *is* $\left\{k, i, q, q^{\delta}\right\}$ *that solve, recursively,*

$$\begin{aligned}
\frac{\beta}{\gamma} &= k\left[f'(k) - 1\right]\frac{1 - f'(k)(1 - \theta_1/2)}{f'(k)(1 - \theta/2) - 1} \\
i &= f'(k) - 1 \\
q &= \frac{\beta}{\gamma}\frac{f'(k)(1 - \theta_0)\,\theta_1/2}{1 - f'(k)(1 - \theta_1/2)} \\
q^{\delta} &= q - kf'(k)\,\theta_1/2
\end{aligned}$$

If $u'(q) = 1$*, the equilibrium with* $\lambda = 0$ *is* $\left\{k, i, q^{\delta}\right\}$ *that solve (10), (28) and (29).*

Lemma 2 *Consider a setting with integrated banking. In the equilibrium with* $\lambda > 0$*,* k *is constant in* θ_0 *and* θ_1*. In the equilibrium with* $\lambda = 0$*,* k *is decreasing in* θ_0 *and in* θ_1*.*

Proposition 1 *Under integrated banking, there is* $\bar{\theta}_0^I > 0$ *such that if* $\theta_0 < \bar{\theta}_0^I$ *there is a unique equilibrium with* $\lambda > 0$ *and if* $\theta_0 \geq \bar{\theta}_0^I$ *there is a unique equilibrium with* $\lambda = 0$*.*

When CBDC is available, buyers can alternatively use CBDC or deposits for their purchases in the goods market. While CBDC is risk-free and unremunerated, deposits are risky and bear interest. If θ_0 is relatively low, the risk of banks' failure is reduced and buyers are willing to deposit their entire money holdings to fully benefit from deposits' expected return. By contrast, if θ_0 is sufficiently high, buyers prefer holding money along with deposits within the period, to partially insure against the risk of bank failure.

4 Fragmented banking

When banking markets are fragmented along national lines, operations between agents and banks located in different countries are not feasible: buyers that wish to deposit and entrepreneurs in need of borrowing are only able to contract with domestic banks. With fragmented banking, the banks' program differs substantially from the one presented in the previous section. To ease the exposition, we do not restate equations or variables that are equivalent to those with integrated banking and simply refer to them in the description of the equilibrium with fragmented banking.

4.1 Settlement market

With a fragmented banking sector, banks in the safe country only lend to safe entrepreneurs, and thus the deposits issued by these banks bear no risk. By contrast, deposits issued by banks in the risky contry are risky because they are backed by risky loans. $W(m, d)$ denotes the value for an agent in the safe country of entering the settlement market with m money balances and d deposits, while $W^{\theta}\left(m^{\theta}, d^{\theta}\right)$ denotes the value for an agent in the risky country of entering the settlement market with m^{θ} money balances and d^θ deposits. As in the case of integrated banking, there is a unique price of money in terms of the numeraire, ϕ, independently of the country where agents trade and regardless of whether the macroeconomic shock occurs or not. x and x^θ denote consumption of the numeraire good by an agent in the safe and the risky country, respectively. $V(m)$ denotes again the value at the beginning of a period for a generic agent who holds m units of money.

The representative agent in the safe country solves

$$W(m,d) = \max_{x,m_{+1}} x + \beta V_{+1}(m_{+1})$$

subject to

$$x + \phi m_{+1} = \phi(1+i_d)d + \phi m + \phi T$$

where i_d is now the interest rate on deposits in the safe country. By inserting the budget constraint in the objective function, we obtain the first order condition on m_{+1}, that we can express one period lagged, as in (3).

The representative agent in the risky country solves

$$W^\theta\left(m^\theta, d^\theta\right) = \max_{x^\theta,m_{+1}} x^\theta + \beta V_{+1}(m_{+1})$$

subject to, in the absence of a macroeconomic shock, with probability $(1-\theta_0)$,

$$x^\theta + \phi m_{+1} = \phi(1+i_d^\theta)d^\theta + \phi m^\theta + \phi T$$

where i_d^θ is the interest rate on deposits in the risky country. If case of failure, with probability θ_0, the budget constraint is instead

$$x^\theta + \phi m_{+1} = \phi\delta d^\theta + \phi^\theta m^\theta + \phi^\theta T$$

By inserting the budget constraints in the objective function, we obtain the first order condition on m_{+1}, that does not depend on the realization of the shock, and that we can express one period lagged as in (3). Again, both the price of money in terms of the numeraire and money demand are the same in the two countries, independently of the current realization of the macroeconomic shock.

The envelope conditions on m give

$$W_m = W_m^\theta = \phi \tag{32}$$

The envelope conditions on d and d^θ are, respectively

$$W_d = \phi\left(1+i_d\right) \tag{33}$$

and

$$W_{d^\theta}^\theta = (1-\theta_0)\,\phi\left(1+i_d^\theta\right) + \theta_0\delta \tag{34}$$

As with integrated banking, the price of money evolves over time according to (5).

4.2 Banking activities

With fragmented banking, the banks' program depends on the realization of the idiosyncratic shock that occurs at the beginning of the period. In the risk-free country, banks solve

$$\max_{d,\ell}\left[(1+i)\,\ell - (1+i_d)\,d + d - \ell\right]$$

subject to $\ell \leq d$. Since in equilibrium the bank's constraint holds with equality, the zero-profit condition for banks in the safe country yields

$$i_d = i \tag{35}$$

In the risky country, banks solve

$$\begin{gathered}\max_{d^\theta,\ell^\theta}(1-\theta_0)\left[\left(1+i^\theta\right)\ell^\theta - \left(1+i_d^\theta\right)d^\theta + d - \ell^\theta\right] \\ +\theta_0\max\langle(1-\theta_1)\left(1+i^\theta\right)\ell^\theta - \left(1+i_d^\theta\right)d^\theta + d - \ell^\theta\rangle^+\end{gathered}$$

subject to $\ell^\theta \leq d^\theta$.

As with the case of integrated banking, with perfectly competitive banks protected by limited liability in equilibrium banks' profits must be zero in the good state. Therefore, taking into account that the bank's constraint binds in equilibrium, we deduce

$$i_d^\theta = i^\theta \tag{36}$$

The banks' liquidation value δ in the risky country is

$$\delta = (1-\theta_1)\left(1+i^\theta\right) \tag{37}$$

since the bank's constraint binds in equilibrium.

4.3 Capital market

The program for suppliers is not affected by the fragmentation of the banking sector since suppliers sell their capital in exchange for money whose price in the settlement market is the same in both countries. Their indifference condition is therefore as in (8).

Entrepreneurs, however, are only able to borrow from domestic banks for their capital investments in this case. In the safe country, entrepreneurs solve

$$\max_{\ell,k} f(k) - (1+i)\phi\ell$$

subject to $\ell \geq p_k k$. Since this constraint holds with equality for $i > 0$, the first-order condition on k is as in (9) and can be rewritten as in (10).

In the risky country, entrepreneurs solve

$$\max_{\ell^\theta,k^\theta} f(k^\theta) - (1+i^\theta)\phi\ell^\theta$$

subject to $\ell^\theta \geq p_k k^\theta$. Using the constraint set to equality, the first-order condition on k^θ is

$$f'(k^\theta) = \phi p_k(1+i^\theta) \tag{38}$$

and using (8) it can be rewritten as

$$f'(k^\theta) = 1 + i^\theta \tag{39}$$

4.4 Goods market

4.4.1 With CBDC

With CBDC, sellers accept to be paid both in deposits and central-bank money in exchange for goods. Since the goods market is perfectly integrated across the borders, sellers can be paid with deposits issued in both countries. Considering a program for sellers similar to the one with integrated banking, we obtain the following sellers' indifference condition regarding the goods sold in exchange for safe deposits (at a price p_d), deposits issued by risky banks in the good state (at a price p_d^θ), deposits issued by risky banks in the bad state (at a price p_d^δ), and money, respectively:

$$\phi p_d\left(1+i_d\right) = 1 \tag{40}$$

$$\phi p_d^\theta\left(1+i_d^\theta\right) = 1 \tag{41}$$

$$\phi p_d^\delta \delta = 1 \tag{42}$$

$$\phi p = 1 \tag{43}$$

In the safe country, buyers deposit all their money holdings, since deposits are risk free and $i_d \geq 0$, and use their deposits to acquire q in the goods market. Therefore $p_d q = d$ and $d = m$ for these buyers.[9]

In the risky country, in the good state buyers acquire q_m^θ with money and q_d^θ with deposits, and their consumption is therefore $q^\theta = q_m^\theta + q_d^\theta$. We anticipate that in the bad state buyers spend their entire money holdings and deposits. Their program at the

[9] In the extreme case where holding money entails no cost $p_d q \leq d$.

beginning of the period is

$$\max_{d^\theta, q_m^\theta, q_d^\theta, q^\delta} \quad (1-\theta_0)\left[u\left(q^\theta\right) + W^\theta\left(m - d^\theta - pq_m^\theta, d^\theta - p_d^\theta q_d^\theta\right)\right] + \theta_0 u\left(q^\delta\right)$$

subject to $d^\theta \le m$, $pq_m^\theta \le m - d^\theta$, $p_d^\theta q_d^\theta \le d^\theta$ and

$$q^\delta = \frac{m - d^\theta}{p} + \frac{d^\theta}{p_d^\delta} \tag{44}$$

Let λ^θ, μ_m^θ and μ_d^θ denote the Lagrange multipliers on the first, second and third constraints. Using (32) and (34), the buyer's first-order conditions on q_m^θ and q_d^θ can be written as

$$(1-\theta_0)\left[u'\left(q^\theta\right) - 1\right] = \mu_m^\theta \tag{45}$$

and

$$(1-\theta_0)\left[u'\left(q^\theta\right) - 1\right] = \mu_d^\theta \tag{46}$$

which implies that $\mu_m^\theta = \mu_d^\theta > 0$ if $u'\left(q^\theta\right) > 1$ and $\mu_m^\theta = \mu_d^\theta = 0$ if $u'\left(q^\theta\right) = 1$. Using (36), (37), (41)-(43), (45) and (46), the buyer's first-order condition on d^θ becomes

$$i^\theta(1-\theta_0)u'\left(q^\theta\right) + \theta_0 u'\left(q^\delta\right)\left[(1-\theta_1)\left(1+i^\theta\right) - 1\right] = \frac{\lambda^\theta}{\phi} \tag{47}$$

The first-order condition on money holdings is

$$\frac{\phi_{-1}}{\beta} = \frac{1}{2}\frac{u'(q)}{p_d} + \frac{1}{2}\left[\lambda^\theta + \frac{\mu_m^\theta}{p} + (1-\theta_0)\phi + \theta_0 u'\left(q^\delta\right)\frac{1}{p}\right]$$

Using (5), (35), (40), (43), (45) and (47), this equation can be rewritten as

$$\frac{\gamma}{\beta} = \frac{1}{2}u'(q)(1+i) + \frac{1}{2}\left[(1-\theta_0)u'\left(q^\theta\right) + \theta_0 u'\left(q^\delta\right)(1-\theta_1)\right]\left(1+i^\theta\right) \tag{48}$$

The deposit constraint for the buyer in the safe country that always holds with equality, $d = m$, combined with the buyer's liquidity constraint to acquire q^δ in the risky country (44), gives

$$q^\delta = \frac{d - d^\theta}{p} + \frac{d^\theta}{p_d^\delta}$$

Combining the banks' and entrepreneurs' constraints with (9), (10), (37)-(39), (42) and (43) yields

$$q^\delta = k + k^\theta\left[(1-\theta_1)f'\left(k^\theta\right) - 1\right] \tag{49}$$

If buyers in the safe country spend all their deposits and $p_d q = d$, using the bank's constraint in the safe country, (10), (35) and (40), we obtain $q = f'(k)\,k$ as in (19). If these buyers are not liquidity constrained, then q satisfies $u'(q) = 1$.

If $\mu_d^\theta > 0$, using the constraints faced by buyers, banks and entrepreneurs in the risky country, as well as (36), (38), (41) and (42), we obtain

$$q^\theta = q^\delta + f'\left(k^\theta\right) k^\theta \theta_1 \tag{50}$$

If $\mu_d^\theta = 0$, then $u'\left(q^\theta\right) = 1$.

If $\lambda^\theta > 0$, $d = d^\theta$. Given the constraints by banks and entrepreneurs, (9), (10), (38) and (39), we deduce that in this case

$$k^\theta = k \tag{51}$$

This implies, from (19) and (49),

$$q^\delta = q\,(1 - \theta_1) \tag{52}$$

and, given (50),

$$q^\theta = q \tag{53}$$

To obtain the value of k in the equilibrium with $\lambda^\theta > 0$, we use (10), (19), (39), (48), (52) and (53) setting $u = \ln$ to obtain

$$k = \frac{\beta}{\gamma} \tag{54}$$

If $u'(q) = u'\left(q^\theta\right) = 1$, for $u = \ln$ (48) can be written as

$$\frac{k}{(1 - \theta_0/2)\, f'(k)\, k + \theta_0/2} = \frac{\beta}{\gamma} \tag{55}$$

If $\lambda^\theta = 0$, from (47) with $u = \ln$ we have

$$q^\delta = \theta_0 q^\theta \frac{\left[1 - (1 - \theta_1)\left(1 + i^\theta\right)\right]}{i^\theta\,(1 - \theta_0)} \tag{56}$$

Using this equation and (49), and given (10) and (39), from (48) we obtain the same value for k as in (54). To obtain k^θ, we combine (56) with (39), (49), (50) and (54) which yields

$$k^\theta \left[f'\left(k^\theta\right) - 1\right] \frac{1 - f'\left(k^\theta\right)(1 - \theta_1)}{f'\left(k^\theta\right)(1 - \theta) - 1} \quad = \quad \frac{\beta}{\gamma} \tag{57}$$

If $u'(q) = u'\left(q^{\theta}\right) = 1$, for $u = \ln$, combining (49) and (56) we obtain

$$k = \left[1 - (1 - \theta_1) f'\left(k^{\theta}\right)\right] \left[\frac{\theta_0}{\left[f'\left(k^{\theta}\right) - 1\right](1 - \theta_0)} + k^{\theta}\right] \tag{58}$$

and, further using (48),

$$\frac{\gamma}{\beta} = \frac{1}{2}\left[f'(k) + \frac{(1 - \theta_0)\,\theta_1 f'\left(k^{\theta}\right)}{1 - (1 - \theta_1) f'\left(k^{\theta}\right)}\right] \tag{59}$$

which solve for k and k^{θ}.

4.4.2 With no CBDC

As with an integrated banking sector, it can be shown that, in the absence of a CBDC, in equilibrium $\lambda^{\theta} > 0$ and buyers deposit all their money holdings for all values of θ_0 and θ_1.[10]

4.5 Equilibrium with CBDC

We can now state the definition of the equilibrium with $\lambda^{\theta} > 0$ under fragmented banking by restating (10), (19), (39) and (51)-(54).

Definition 3 *An equilibrium with $\lambda^{\theta} > 0$ is $\left\{q, q^{\theta}, q^{\delta}, i, i^{\theta}, k, k^{\theta}\right\}$ that solve, recursively,*

$$\begin{aligned}
k &= k^{\theta} = \frac{\beta}{\gamma} \\
i &= i^{\theta} = f'(k) - 1 \\
q &= q^{\theta} = f'(k)\,k \\
q^{\delta} &= f'(k)\,k\,(1 - \theta_1)
\end{aligned}$$

If $u'\left(q^{\theta}\right) = u'(q) = 1$, the equilibrium is $\left\{q^{\delta}, i, i^{\theta}, k, k^{\theta}\right\}$ that solve (10), (39), (51), (52) and (55).

We can also state the definition of the equilibrium with $\lambda^{\theta} = 0$ under fragmented banking by replicating (10), (19), (39), (49), (50), (54) and (57).

[10] The details are relegated to the appendix.

Definition 4 *An equilibrium with* $\lambda^\theta = 0$ *is* $\left\{q, q^\theta, q^\delta, i, i^\theta, k, k^\theta\right\}$ *that solve, recursively,*

$$
\begin{aligned}
k &= \frac{\beta}{\gamma} \\
k^\theta &= \frac{\beta}{\gamma} \frac{f'\left(k^\theta\right)(1-\theta)-1}{\left[f'\left(k^\theta\right)-1\right]\left[1-f'\left(k^\theta\right)(1-\theta_1)\right]} \\
i &= f'(k) - 1 \\
i^\theta &= f'\left(k^\theta\right) - 1 \\
q &= f'(k)\, k \\
q^\delta &= k + k^\theta\left[(1-\theta_1) f'\left(k^\theta\right) - 1\right] \\
q^\theta &= q^\delta + f'\left(k^\theta\right) k^\theta \theta_1
\end{aligned}
$$

If $u'\left(q^\theta\right) = u'(q) = 1$*, the equilibrium is* $\left\{q^\delta, i, i^\theta, k, k^\theta\right\}$ *that solve (10), (39), (49), (58) and (59).*[11]

Comparing the equilibrium with $\lambda > 0$ under integrated banking and the one with $\lambda^\theta > 0$ under fragmented banking, we notice that capital allocation is identical in both cases.[12] Moreover, in both equilibria the level of capital invested by risky and safe entrepreneurs is the same. On the contrary, if we consider the equilibria with slack deposit constraints, with $\lambda = 0$ under integrated banking and $\lambda^\theta = 0$ under fragmented banking, capital allocation differs in these two cases. Whereas with integrated banks safe and risky entrepreneurs still invest the same amount of capital, this is not the case with fragmented banks. With an integrated banking sector all banks, regardless of their location, hold identical portfolios. Instead, with fragmentation in the banking sector, the volume of bank lending in the risky country is smaller than the one in the safe country, as depositors in the safe country deposit all their money holdings and those in the risky country deposit only a fraction of them implying some disintermediation. Thus, a difference emerges between fragmented banks across countries, that does not arise under integrated banking.

Lemma 3 *Let* $u = \ln$ *and consider the setting with fragmented banking. In the equilibrium with* $\lambda^\theta > 0$*,* k *and* k^θ *are equal and constant in* θ_0 *and* θ_1*. In the equilibrium with* $\lambda^\theta = 0$*,* k *is constant in* θ_0 *and* θ_1 *and* k^θ *is decreasing in* θ_0 *and* θ_1*.*

Proposition 2 *Under fragmented banking, there is* $0 < \bar{\theta}_0^F$ *such that if* $\theta_0 < \bar{\theta}_0^F$ *there is a unique equilibrium with* $\lambda^\theta > 0$ *and if* $\theta_0 \geq \bar{\theta}_0^F$ *there is a unique equilibrium with* $\lambda^\theta = 0$*.*

As with integrated banking, there is a threshold value of the probability that the bad state occurs, denoted as $\bar{\theta}_0^F$, above which agents that have the choice between risky

[11] We present the cases $u'\left(q^\theta\right) > u'(q) = 1$ and $u'(q) > u'\left(q^\theta\right) = 1$ in the Appendix.

[12] With less risk averse agents, with fragmented banking the level of invested capital in the safe country would be higher than with integrated banking, since under integrated banking banks in the safe country are risky.

deposits and risk-free central bank money are not willing to deposit their entire money holdings and prefer holding a portfolio of both means of payment within the period. Unlike the case of integrated banking, with fragmented banking the deposit constraint is slack only for agents located in the risky country. Agents in the safe country stricly prefer deposits than money, as deposits bear interet and are equally safe.

Proposition 3 *1. For any θ_1, $\bar{\theta}_0^F < \bar{\theta}_0^I$.*

2. $\bar{\theta}_0^F$ and $\bar{\theta}_0^I$ are decreasing in θ_1.

3. Under fragmented banking, there always exist combinations $\{\theta_0, \theta_1\}$ such that CBDC matters (i.e. $\lambda^\theta = 0$ in equilibrium), while there are conditions on f such that under integrated banking CBDC is irrelevant ($\lambda > 0$) for all $\{\theta_0, \theta_1\}$.

Proposition 3 states that the threshold value of θ_0 above which all deposit contraints cease to be binding is lower when the banking sector is fragmented than when it is integrated. With integrated banking, deposits are risky, as they are for agents in the risky country under fragmented banking. However, a main difference between the two settings is the degree of diversification of banks' loans portfolios. Integrated banks are exposed to safe and risky loans, while fragmented banks in the risky country only hold risky loans. Therefore, for a same probability of banks' failure, θ_0, losses borne by deposits in case the bad macroeconomic shock realizes are greater under fragmented banking, as a comparison between (7) and (37) suggests.

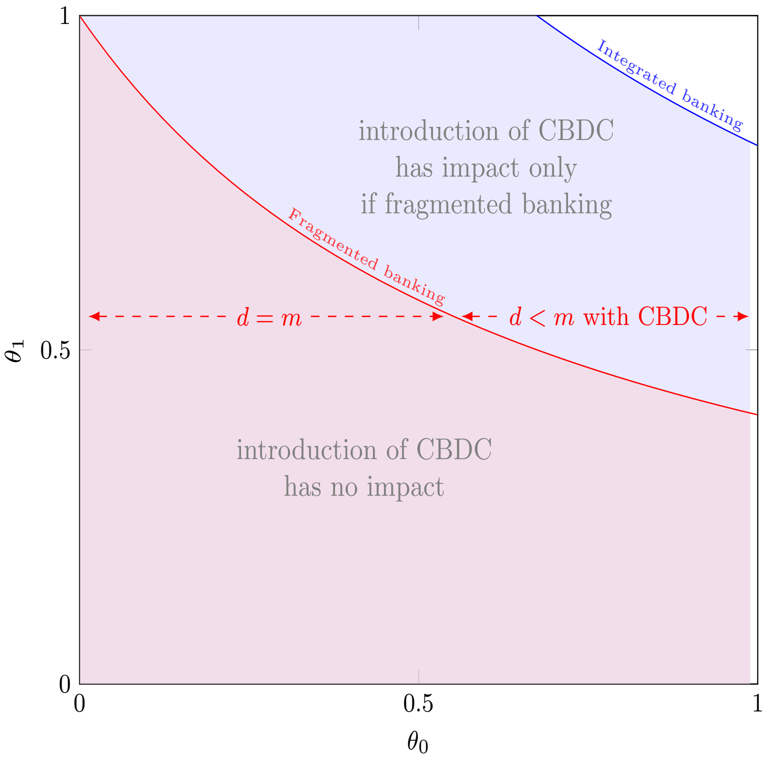

Figure 2. Equilibrium: integrated banking vs fragmented banking

The result stated in Proposition 3 is illustrated in Figure 2, that shows that there is a range of combinations of values of θ_0 and θ_1 for which $\lambda^\theta = 0$ under fragmented banking and $\lambda > 0$ under integrated banking. As the second part of Proposition 3 states, it could be that the equilibrium with $\lambda = 0$ in integrated banking fails to exist. Depending on the function f, agents may be willing to deposit their entire money holdings with banks even if θ_0 and θ_1 are close to 1, if the return on safe loans ensures an attractive remuneration on deposits. With fragmented banking, instead, the equilibrium with $\lambda^\theta = 0$ always exists since in the risky country deposits are backed only by risky loans and this leads depositors to unambiguously prefer a portfolio of deposits and CBDC if the latter is available and θ_0 and θ_1 are sufficiently high.

Lemma 4 *Let $\theta_0 \geq \bar{\theta}_0^F$. Capital allocation satisfies $k/k^\theta = 1$ in integrated banking and $k/k^\theta > 1$ in fragmented banking.*

As Lemma 4 indicates, when a CBDC is introduced and adopted under fragmented banking, the ratio of capital investment levels across countries is closer to the optimum, stated in (1) and (2), than under integrated banking.

5 Welfare effect of CBDC

From the discussion in Sections 3 and 4, we conclude that, for sufficiently low levels of macroeconomic risk (i.e., for $\theta_0 \leq \bar{\theta}_0^F$), the equilibrium allocation would not be affected by the introduction of a CBDC, either with integrated or fragmented banking. Since deposits are not that risky in this case, an unremunerated CBDC is not attractive enough and does not alter agents' portfolio choice. In this section we assess the effect of introducing CBDC for moderate levels of macroeconomic risk in the context of our model; i.e., for levels of θ_0 that satisfy $\bar{\theta}_0^F \leq \theta_0 \leq \bar{\theta}_0^I$. Recall that, for the two types of banking sector considered, if the central bank issues physical cash and there is no digital central-bank money, the equilibrium necessarily displays binding deposit constraints since deposits are the only means of payment suitable for transactions in the goods market. Thus, with no CBDC, we have $\lambda > 0$ under integrated banking and $\lambda^\theta > 0$ under fragmented banking, regardless of the value of θ_0. If the central bank issues a digital currency, agents may partially substitute deposits for this central-bank digital currency.

From Proposition 3, $\bar{\theta}_0^F < \bar{\theta}_0^I$ and therefore, introducing CBDC for a value of θ_0 higher than, but close to, $\bar{\theta}_0^F$ would leave the allocation unaffected in the case of integrated banking. Even with an available risk-free option provided by the central bank, depositors would strictly prefer deposits to central-bank money as deposits are remunerated and not too risky. More precisely, with integrated banking, the introduction of CBDC has no effect on welfare for $\theta_0 \in \left[\bar{\theta}_0^F, \bar{\theta}_0^I\right]$ since the equilibrium has $\lambda > 0$ over this range of values of θ_0. However, the introduction of a CBDC would affect the allocation under fragmented banking (since the equilibrium with $\lambda^\theta > 0$ would switch to one with $\lambda^\theta = 0$).

With fragmented banking, welfare is

$$\begin{aligned}\mathcal{W} &= u(q) - q + (1-\theta_0)\left[u\left(q^\theta\right) - q^\theta\right] + \theta_0\left[u\left(q^\delta\right) - q^\delta\right] \\ &+ f(k) + (1-\theta_0\theta_1) f\left(k^\theta\right) - k - k^\theta\end{aligned}$$

Proposition 4 *The introduction of CBDC improves welfare under fragmented banking and has no welfare effect under integrated banking for* $\theta_0 \in \left[\bar{\theta}_0^F, \bar{\theta}_0^I\right]$.

In order to prove the welfare effect of a CBDC under fragmented banking, we proceed as follows. First, we compute the effect on welfare $\mathcal{W}$ of an increase in θ_0 by assuming that the CBDC is not available. This means that agents can only use bank deposits for their transactions in the goods market and therefore they deposit their entire money holdings with banks, irrespective of the value of θ_0. The relevant equilibrium equations are therefore those that correspond to the equilibrium with $\lambda^\theta > 0$. Second, we compute the effect on welfare $\mathcal{W}$ of an increase in θ_0 by assuming that the central bank issues a CBDC. When a CBDC is available at these levels of risk, for $\theta_0 \geq \bar{\theta}_0^F$, buyers in the risky country choose to diversify their means of payment by holding both risky deposits and risk-free central-bank money. Therefore the deposit constraint is slack for these agents, with $\lambda^\theta = 0$. Finally, we compare the respective derivatives of $\mathcal{W}$ with respect to θ_0 for $\theta \geq \bar{\theta}_0^F$, if a CBDC is available and in the absence of a CBDC. We can show that both derivatives are negative: intuitively, an increase in the probability of a bad macroeconomic shock worsens welfare.

Let $\mathcal{W}^N$ denote welfare when a CBDC is not available and $\mathcal{W}^C$ denote welfare when a CBDC is issued by the central bank. The derivative of $\mathcal{W}^N$ with respect to θ_0 is

$$\frac{d\mathcal{W}^N}{d\theta_0} = -\left[u\left(q^\theta\right) - q^\theta\right] + u\left(q^\delta\right) - q^\delta - \theta_1 f\left(k^\theta\right)$$

which is negative. Therefore, in the absence of a CBDC the (only) effect of an increase in θ_0 is given by the reduction in trade surplus between buyers and sellers and the lower expected output produced by entrepreneurs.

The derivative of $\mathcal{W}^C$ with respect to θ_0 is

$$\begin{aligned}\frac{d\mathcal{W}^C}{d\theta_0} &= (1-\theta_0)\left[u'\left(q^\theta\right) - 1\right]\frac{dq^\theta}{d\theta_0} + \theta_0\left[u'\left(q^\delta\right) - 1\right]\frac{dq^\delta}{d\theta_0} + \left[(1-\theta_0\theta_1) f'\left(k^\theta\right) - 1\right]\frac{dk^\theta}{d\theta_0} \\ &+ \frac{d\mathcal{W}^N}{d\theta_0}\end{aligned}$$

With an available CBDC, the effect of an increase in θ_0 found for the case without a CBDC is also present. However, the increase in θ_0 has additional effects. In the Appendix, we develop the derivative of $\mathcal{W}^C$ with respect to θ_0 in order to determine the sign of the difference between this derivative and the derivative of $\mathcal{W}^\mathcal{N}$ with respect to θ_0. We show that the derivative of $\mathcal{W}^C$ with respect to θ_0 is higher than the derivative of $\mathcal{W}^N$ with respect to θ_0. Therefore, since welfare is identical for $\theta_0 = \bar{\theta}_0^F$ whether CBDC exists or not, the level of welfare is higher with CBDC for $\theta_0 \in \left[\bar{\theta}_0^F, \bar{\theta}_0^I\right]$ as stated in Proposition 4. This is illustrated in Figure 3 that depicts both $\mathcal{W}^\mathcal{N}$ and $\mathcal{W}^C$ as functions of θ_0. The introduction of a CBDC has two important effects that ultimately affect welfare. On the one hand, the CBDC allows buyers to partially insure against deposits' risk, thereby smoothing consumption. On the other, it modifies capital allocation across countries by increasing the ratio k/k^θ, which is unambiguously welfare improving for sufficiently high levels of θ_0.

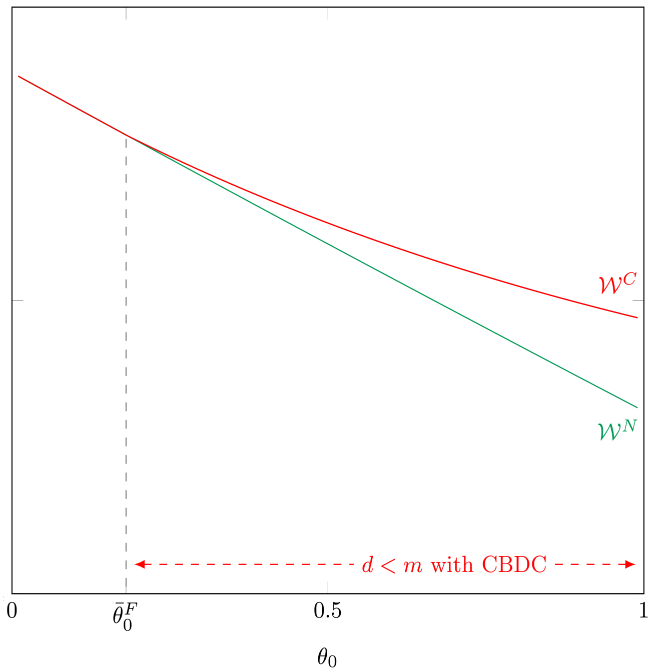

Figure 3. Welfare under fragmented banking: with and without CBDC

6 Deposit insurance

In the context of a monetary union, and given imperfectly integrated banking markets across the borders, issuing a CBDC is a way of providing a uniform digital means of payment for all states. Establishing a federal deposit insurance would be an alternative tool to achieve a similar objective. In this section, we introduce a deposit insurance into the model to compare its welfare effects with those of a CBDC.

To model deposit insurance, we assume that a (nominal) tax τ is collected on buyers in the SM in $t-1$ (and unused resources in t are rebated to buyers during the SM). We assume that only a fraction $\chi \leq 1$ can be shared across countries and the remaining fraction can only be used at the country level. Thus $\chi = 1$ corresponds to a common deposit insurance scheme, and $\chi = 0$ to a purely national deposit insurance scheme. The collected funds are transferred to deposit holders at redemption, during the SM in t, in case of bank failure. In what follows we present the relevant equations with deposit insurance in the setting with fragmented banking (we omit the capital market which is unaffected).

6.1 Fragmented banking with no CBDC

Settlement market

In the settlement market, the program is as without deposit insurance except for the fact that buyers must afford the tax τ and receive a transfer in case of unused deposit insurance funds collected in $t-1$. In the absence of a macro shock, with probability $(1-\theta_0)$, buyers in both countries receive a transfer equal to τ_{-1}. If the macro shock occurs, with probability θ_0, buyers in the risky country receive no transfer and those in the safe country receive $(1-\chi)\,\tau_{-1}$ since a fraction χ of the collected funds are transferred to the risky country in that case.

Banking activities

The program for banks in the risk-free country is as in the absence of deposit insurance, and hence $i_d = i$. Whether deposits are fully or partially insured, the program for banks in the risky country is also as in the case with no deposit insurance since banks only care about the good state, and thus $i_d^\theta = i^\theta$. The difference lies in the liquidation value of deposits, which is now δ^{DI}.

The banks' liquidation value δ^{DI} in the risky country (per unit of deposit) is

$$\delta^{DI} = \frac{(1-\theta_1)\left(1+i^\theta\right)\ell^\theta + (1+\chi)\,\tau_{-1}}{d^\theta}$$

since the country hit by the macro shock uses the totality of its own deposit insurance funds and receives a fraction χ of those collected by the other country.

Let $\tau_{-1} = \tilde{\tau}\ell^\theta\left(1+i^\theta\right)$. Since the bank's constraint binds in equilibrium, we obtain

$$\delta^{DI} = \left[1-\theta_1+(1+\chi)\,\tilde{\tau}\right]\left(1+i^\theta\right) \tag{60}$$

Goods market

The buyer's first-order condition on d^θ is

$$i^\theta\,(1-\theta_0)\,u'\left(q^\theta\right) + \theta_0 u'\left(q^\delta\right)\left\{\left[1-\theta_1+(1+\chi)\,\tilde{\tau}\right]\left(1+i^\theta\right)-1\right\} \;=\; \frac{\lambda^\theta}{\phi}$$

thus slightly different from (47) because of the difference in the liquidation value of deposits. With a procedure similar to the one used for the setting with no deposit insurance, we find that q solves (19) and $q^\theta = q$, as in the case without deposit insurance, and q^δ solves

$$q^\delta \quad = \quad f'\left(k^\theta\right) k^\theta \left[1 - \theta_1 + (1+\chi)\,\tilde{\tau}\right] \tag{61}$$

Taking into account the value of δ^{DI}, the first-order condition on money holdings is

$$\frac{\gamma}{\beta} = \frac{1}{2} u'(q)(1+i) + \frac{1}{2}\left[(1-\theta_0)\,u'\left(q^\theta\right) + \theta_0 u'\left(q^\delta\right)\left[1-\theta_1+(1+\chi)\,\tilde{\tau}\right]\right]\left(1+i^\theta\right)$$

With $u = \ln$, and using (10), (19), (39), (53) and (61), we find that $k = k^\theta$ given by (54) which implies that $i = i^\theta$ and $q = q^\theta$.

We can now state the definition of the equilibrium in fragmented banking with deposit insurance and no CBDC, where $\lambda, \lambda^\theta > 0$ for all parameter values, for $u = \ln$ and the case $u'(q) > 1$.

Definition 5 *An equilibrium in fragmented banking with deposit insurance is* $\left\{q, q^\theta, q^\delta, i, i^\theta, k, k^\theta\right\}$ *that solve, recursively, (54), (51), (10), (39), (19), (53) and (61).*

Hence with deposit insurance welfare is

$$\begin{aligned}
\mathcal{W}^{\mathcal{DI}} \quad &= \quad (2-\theta_0)\left[u(q) - q\right] + \theta_0\left[u\left(q^{\delta,DI}\right) - q^{\delta,DI}\right] - 2k \\
&+ \quad (2-\theta_0\theta_1)\, f(k) - 2(\gamma-1)\,\tilde{\tau} k f'(k)
\end{aligned}$$

where DI stands for "deposit insurance" and the notation $q^{\delta,DI}$ indicates that q^δ may differ with deposit insurance and the last term is the lump-sum tax to fund the deposit insurance. If $\tilde{\tau} = 0$, then $q^{\delta,DI} = q^\delta$ and the expression for welfare boils down to the one with no deposit insurance (and no CBDC). In the other extreme, with full deposit insurance, $\tilde{\tau}$ is such that $q^{\delta,DI} = q$. With $q^{\delta,DI} = q\left[1-\theta_1+(1+\chi)\,\tilde{\tau}\right]$, this requires $(1+\chi)\,\tilde{\tau} = \theta_1$. In this case, the condition for $\mathcal{W}^{\mathcal{N}} < \mathcal{W}^{DI}$ is

$$\begin{aligned}
&(2-\theta_0)\left[u(q) - q\right] + \theta_0\left[u\left(q^{\delta,N}\right) - q^{\delta,N}\right] - 2k + (2-\theta_0\theta_1)\, f(k) \\
< \quad &2\left[u(q) - q\right] - 2k + (2-\theta_0\theta_1)\, f(k) - 2(\gamma-1)\,\tilde{\tau} k f'(k)
\end{aligned}$$

Simplifying, and considering that in this case $(1+\chi)\,\tilde{\tau} = \theta_1$, we obtain

$$\theta_0\left\{u(q) - q - \left[u\left(q^{\delta,N}\right) - q^{\delta,N}\right]\right\} \quad > \quad \frac{2(\gamma-1)\,\theta_1}{(1+\chi)} k f'(k)$$

Using the definition of the equilibrium with deposit insurance and $u = \ln$, we can define

θ_0^{DI}, the threshold value of θ_0 above which $\mathcal{W}^{DI} > \mathcal{W}^{\mathcal{N}}$ which is

$$\theta_0^{DI} \quad = \quad \frac{2\left(\gamma-1\right)kf'\left(k\right)}{1+\chi}\frac{-1}{\frac{\ln(1-\theta_1)}{\theta_1}+q} \tag{62}$$

where k and q are independent of θ_0 and θ_1.

Proposition 5 *For $\gamma > 1$,*

$$\theta_0^{DI} > 0, \quad \frac{\partial\theta_0^{DI}}{\partial\theta_1} < 0, \quad \lim_{\theta_1\to 0^+}\theta_0^{DI} = \frac{2\left(\gamma-1\right)kf'\left(k\right)}{\left(1+\chi\right)}\frac{1}{1-q}, \quad \lim_{\theta_1\to 1^-}\theta_0^{DI} = 0$$

and full DI dominates no DI for $\theta_0 > \theta_0^{DI}$.

Note also that for any θ_0

$$\frac{\partial\mathcal{W}^{DI}}{\partial\theta_0} = -\theta_1 f\left(k\right) > \frac{\partial\mathcal{W}^{\mathcal{N}}}{\partial\theta_0} = -\left[u\left(q\right)-q\right] + u\left(q^{\delta}\right) - q^{\delta} - \theta_1 f\left(k\right) \tag{63}$$

We now compare the welfare levels that correspond to two alternative scenarios with fragmented banking: one with a deposit insurance scheme and no CBDC, and another one with CBDC and no deposit insurance. In the following proposition $\mathcal{W}^{DI}$ denotes welfare with full deposit insurance (and without CBDC) and $\mathcal{W}^{C}$ denotes welfare with CBDC (and without deposit insurance).

Proposition 6 *Let $u = \ln$. With fragmented banking, given β, γ and χ, there exists $0 < \hat{\theta}_1 < 1$ such that*

$\bar{\theta}_0^F > \theta_0^{DI}$ *for* $\theta_1 < \hat{\theta}_1$

$\bar{\theta}_0^F < \theta_0^{DI}$ *for* $\theta_1 > \hat{\theta}_1$

Hence, i) if $\theta_1 < \hat{\theta}_1$, $\mathcal{W}^{DI} > \mathcal{W}^{C}$ for $\theta_0 \in \left(\theta_0^{DI}, \bar{\theta}_0^F\right)$ and ii) if $\theta_1 > \hat{\theta}_1$, $\mathcal{W}^{C} > \mathcal{W}^{DI}$ for $\theta_0 \in \left(\bar{\theta}_0^F, \theta_0^{DI}\right)$.

From Proposition 6, a federal deposit insurance allows achieving higher social welfare than a CBDC if the output loss due to a macroeconomic shock is relatively small, for a given level of χ. If instead macroeconomic shocks are relatively severe, a CBDC is a superior policy for welfare. In particular, there always exists a region where a CBDC dominates full deposit insurance, as illustrated in Figure 4. The intuition is as follows. With deposit insurance, depositors in the risky country bring all their funds to the banks since the potential losses are covered by the deposit insurance scheme (up to a certain point, given by the parameter χ). Since banks in the risky country do not take into account the losses in case the bad state realizes, they lend to firms the same amount as if they were safe. The result is a suboptimal capital allocation, since the level of capital is k in both the safe and the risky country. When θ_1 is not that high, the social cost of this capital allocation is limited because the optimal level of k^{θ} is not that small. The benefit of deposit insurance stemming from improved risk sharing among depositors offsets the inefficiency in capital investment in that case. However, the opposite is true for high levels of θ_1 since then the difference between the optimal level of k^{θ} and the equilibrium one is sizable.

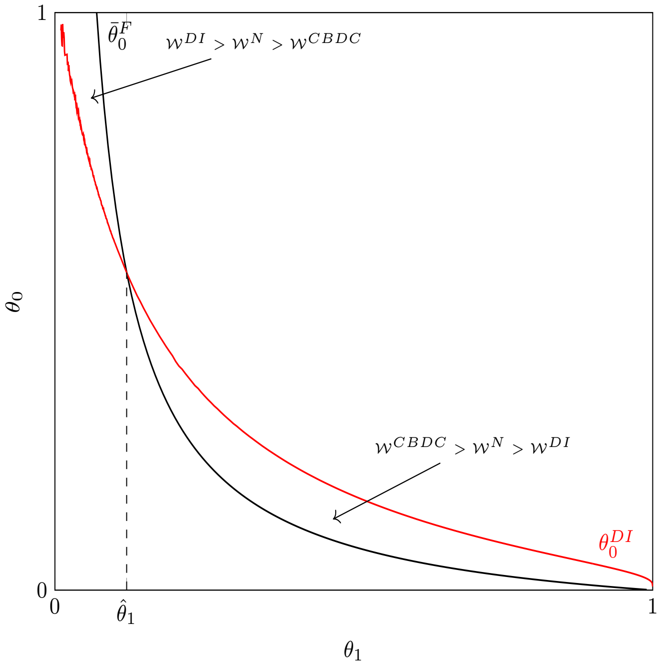

Figure 4. Welfare under fragmented banking: with CBDC or with DI

6.2 Fragmented banking with CBDC and deposit insurance

Finally, we consider a configuration with fragmented banking where both CBDC and the federal deposit insurance are implemented. With binding deposit constraints in both countries, the definition of the equilibrium is as the one with deposit insurance and without CBDC. However, relative to the setting with deposit insurance described earlier, when a CBDC is introduced the equilibrium may display a slack deposit constraint in the risky country. We state the definition of both equilibria as follows.

Definition 6 *An equilibrium in fragmented banking with deposit insurance and CBDC with* $\lambda^{\theta} > 0$ *is* $\left\{q, q^{\theta}, q^{\delta}, i, i^{\theta}, k, k^{\theta}\right\}$ *that solve, recursively, (54), (51), (10), (39), (19), (53) and (61).*

Definition 7 *An equilibrium in fragmented banking with CBDC and deposit insurance, with* $\lambda^{\theta} = 0$ *is* $\left\{q, q^{\theta}, q^{\delta}, i, i^{\theta}, k, k^{\theta}\right\}$ *that solve, recursively, (54), (10), (39), (19) and*

$$\begin{aligned}
\frac{\beta}{\gamma} &= k^{\theta}\left[f'\left(k^{\theta}\right)-1\right]\frac{1-f'\left(k^{\theta}\right)\left[1-\theta_1+(1+\chi)\tilde{\tau}\right]}{f'\left(k^{\theta}\right)\left[1-\theta+\theta_0(1+\chi)\tilde{\tau}\right]-1} \\
q^{\delta} &= k+k^{\theta}\left\{f'\left(k^{\theta}\right)\left[1-\theta_1+(1+\chi)\tilde{\tau}\right]-1\right\} \\
q^{\theta} &= q^{\delta}+f'\left(k^{\theta}\right)k^{\theta}\left[\theta_1-(1+\chi)\tilde{\tau}\right]
\end{aligned}$$

Note that the equilibrium with λ^θ is as in the case with CBDC and no deposit insurance, except that θ_1 is replaced by $\theta_1 - (1+\chi)\,\tilde{\tau}$ in the equations for k^θ, q^θ and q^δ. Since θ_1 does not affect the other equilibrium equations, we deduce that this equilibrium is equivalent to the one with CBDC and no deposit insurance with a lower value of θ_1.

7 Conclusion

We have presented a model of a monetary union to study the effects of a central-bank digital currency under two alternative contexts, one with integrated banking markets across the countries of the union and the other with fragmented banking markets. The analysis of dimensions like consumption risk sharing and resource allocation across countries, specific to a monetary union, leads to policy implications that had not been the focus of previous work on CBDC. A CBDC may lead to higher welfare when banking markets are fragmented across the countries of the union. Implementing a federal deposit insurance when markets are fragmented is also welfare improving, but the welfare improvement stemming from CBDC may be higher if the countries of the union depending on the severity of macroeconomic shocks. For a comprehensive comparison between these two supranational policies, the relative costs of these two policies as well as their feasibility in terms of commitment power and incentives of the different actors are crucial. These aspects are left for future research.

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Appendix

Proofs

Proof of Lemma 2

In the equilibrium with $\lambda > 0$, from (24) k is constant in θ_0 and θ_1. In the equilibrium with $\lambda = 0$, differentiating (27) with respect to θ_0 we obtain

$$
\begin{aligned}
&-f'(k)\theta_1/2 + f''(k)\left(1-\theta/2\right)\frac{dk}{d\theta_0} \\
=\ &\frac{\gamma}{\beta}\left\{\left[f'\left(k\right)-1\right]\left[1-f'(k)\left(1-\theta_1/2\right)\right]+kf''\left(k\right)\left[2-2f'(k)\left(1-\theta_1/2\right)-\theta_1/2\right]\right\}\frac{dk}{d\theta_0}
\end{aligned}
$$

Therefore a sufficient condition for $dk/d\theta_0$ to be negative is

$$\frac{\gamma}{\beta}k\left[2-2f'(k)\left(1-\theta_1/2\right)-\theta_1/2\right]-\left(1-\theta/2\right)\leq 0$$

Using (27), this condition can be rewritten as

$$-\left[f'(k)-1\right]^2\left(1-\theta_1/2\right)+\left\{\left[f'(k)\right]^2\left(1-\theta_1/2\right)-1\right\}\theta/2\leq 0$$

which always holds since in equilibrium $f'(k)\left(1-\theta_1/2\right) < 1 < f'(k)$.

Differentiating (27) with respect to θ_1 we obtain

$$
\begin{aligned}
&-f'(k)\theta_0/2 - \frac{\gamma}{\beta}k\left[f'\left(k\right)-1\right]f'(k)\left(1/2\right) + f''(k)\left(1-\theta/2\right)\frac{dk}{d\theta_1} \\
=\ &\frac{\gamma}{\beta}\left\{\left[f'\left(k\right)-1\right]\left[1-f'(k)\left(1-\theta_1/2\right)\right]+kf''\left(k\right)\left[2-2f'(k)\left(1-\theta_1/2\right)-\theta_1/2\right]\right\}\frac{dk}{d\theta_1}
\end{aligned}
$$

Hence the condition for $\frac{dk}{d\theta_1} < 0$ is the same as for $\frac{dk}{d\theta_0} < 0$ and therefore we conclude that $\frac{dk}{d\theta_1} < 0$.

Proof of Proposition 1

From (16), the Lagrange multiplier on the deposit constraint λ is given by

$$\frac{\lambda}{\phi} = \left(1-\theta_0\right)u'\left(q\right)i + \theta_0 u'\left(q^\delta\right)\left[\left(1-\theta_1/2\right)\left(1+i\right)-1\right]$$

Using (10), (18) and (19) and setting $u = \ln$, this can be rewritten as

$$\frac{\lambda}{\phi} = \frac{1}{kf'(k)}\left[f'(k)-1-\frac{\theta_0\theta_1/2}{1-\theta_1/2}\right]$$

which implies that $\lambda > 0$ for $\theta_0 = 0$. Since from (24) k is independent of θ_0, λ/ϕ is decreasing in θ_0 and we deduce that there is a value $\bar{\theta}_0^I$ such that for $\theta_0 \geq \bar{\theta}_0^I$ the multiplier λ equals zero.

From (10), (18), (19) and (24), k, i, q and q^{δ} are unique in the equilibrium with $\lambda > 0$. To show unicity in the equilibrium with $\lambda = 0$, we rewrite (27) as

$$k\left[f'(k)-1\right]=\frac{\beta}{\gamma}\frac{f'(k)\left(1-\theta_0\theta_1/2\right)-1}{1-f'(k)\left(1-\theta_1/2\right)} \tag{64}$$

Let $\underline{k}(\theta_0,\theta_1)$ and $\bar{k}(\theta_0,\theta_1)$ given by

$$f'\left(\underline{k}(\theta_0,\theta_1)\right)=\frac{1}{1-\theta_1/2},\qquad f'\left(\bar{k}(\theta_0,\theta_1)\right)=\frac{1}{1-\theta_0\theta_1/2} \tag{65}$$

We have $\underline{k}(\theta_0,\theta_1)\leq\bar{k}(\theta_0,\theta_1)\leq k^{*}$.

Equilibrium requires both sides of (64) to be positive. The left side is positive iff $0\leq k\leq k^{*}$, while the right side is positive iff $\underline{k}(\theta_0,\theta_1)\leq k\leq\bar{k}(\theta_0,\theta_1)$. We thus look for a solution in $\left[\underline{k},\bar{k}\right]$. Note that the right side is continuous and strictly decreasing on $(\underline{k},k^{*}]$, from $+\infty$ to $-\frac{\beta}{\gamma}\theta_0$. Hence, there exists at least such one solution; and it belongs to $\left[\underline{k},\bar{k}\right]$.

k solves $G(k)=0$ with

$$G(x)\quad\equiv\quad\frac{\beta}{\gamma}\times\frac{\left(1-\theta_0\theta_1/2\right)f'(x)-1}{1-\left(1-\theta_1/2\right)f'(x)}-\left(f'(x)-1\right)x \tag{66}$$

We claim that (with some assumptions on f) G has a unique solution in $\left(\underline{k},\bar{k}\right]$, and that this solution satisfies $G'<0$. We rewrite $G(x)$ as

$$\begin{aligned}G(x)&=\frac{\beta}{\gamma}\times\left(-\frac{1-\theta_0\theta_1/2}{1-\theta_1/2}+\frac{\left(\theta_1/2\right)\left(1-\theta_0\right)}{1-\theta_1/2}\frac{1}{1-\left(1-\theta_1/2\right)f'(x)}\right)-\left(f'(x)-1\right)x\\&\equiv h\left(f'(x)\right)-\left(f'(x)-1\right)x\end{aligned} \tag{67}$$

Note that the function h is C^2 on $\left(\frac{1}{1-\theta_1},+\infty\right)$, with $h'>0$, $h''>0$. Derivation of (67) yields

$$G'(x)=f''(x)\,h'\left(f'(x)\right)-\left(xf''(x)+f'(x)-1\right) \tag{68}$$

$$G''(x)=f'''(x)\,h'\left(f'(x)\right)+\left(f''\right)^2(x)\,h''\left(f'(x)\right)-\left(2f''(x)+xf'''(x)\right) \tag{69}$$

The following additional assumption on f is a sufficient condition for our purpose:

Assumption 1 *$f'''>0$ and $2f''(x)+xf'''(x)<0$*

(The latter condition is equivalent to $\left(f'(x)-1\right)x$ being concave.)

Note that assumption 1 is satisfied for $f(k)=Ak^{a}$ with $0<a<1$ which we assumed.

Under assumption 1, $G''>0$ (G is strictly convex). Being C^2 on $(\underline{k},+\infty)$, there are at most two solutions to $G(x)=0$ on this interval. (Implied by strict convexity : take the two extreme roots, say $x_1<x_2$, then for any $x_1<x<x_2$, $G(x)=G\left(\frac{x_2-x}{x_2-x_1}x_1+\frac{x-x_1}{x_2-x_1}x_2\right)<\frac{x_2-x}{x_2-x_1}G(x_1)+\frac{x-x_1}{x_2-x_1}G(x_2)=0$.)

Now, $\lim_{x\longrightarrow\left(\frac{1}{1-\theta_1}\right)^+} = +\infty$, $G(k^*) = -\theta_0\frac{\beta}{\gamma} < 0$, and $\lim_{x\longrightarrow+\infty} = +\infty$. Hence, there is one root/solution in $(\underline{k}, k^*)$ and one in $(k^*, +\infty)$. In addition, one can easily see that $G' < 0$ for the lowest solution, and $G' > 0$ for the largest (see figure below).

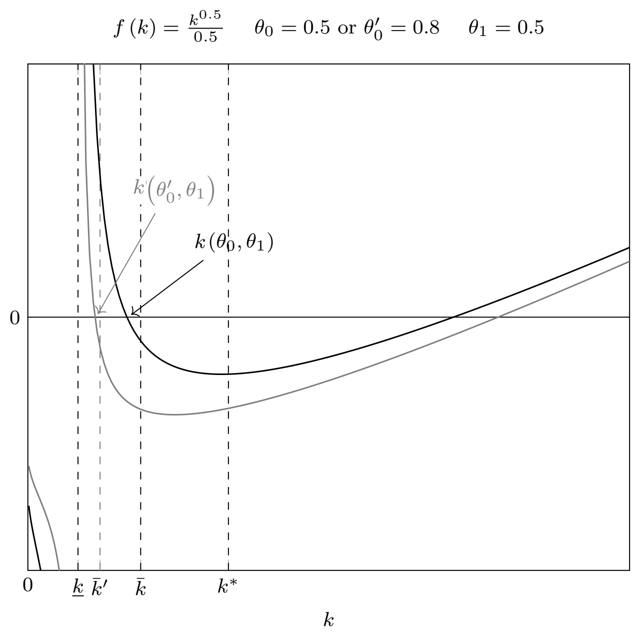

Figure A1. The function $G(.)$ and its roots

This shows that k exists, is unique, with $G'(k) < 0$.

Proof of Lemma 3

In the equilibrium with $\lambda^\theta > 0$, from (51) $k^\theta = k$ and from (54) k is constant in θ_0 and θ_1. In the equilibrium with $\lambda^\theta = 0$, from (54) k is again constant in θ_0 and θ_1. From (57) $dk^\theta/d\theta_0$ is given by

$$\frac{\gamma}{\beta}\left\{\left[2 - 2(1-\theta_1) f'\left(k^\theta\right) - \theta_1\right] f''\left(k^\theta\right) k^\theta + \left[f'\left(k^\theta\right) - 1\right]\left[1 - (1-\theta_1) f'\left(k^\theta\right)\right]\right\}\frac{dk^\theta}{d\theta_0}$$
$$= f''\left(k^\theta\right)(1-\theta)\frac{dk^\theta}{d\theta_0} - \theta_1 f'\left(k^\theta\right) \quad (70)$$

A sufficient condition for $dk^\theta/d\theta_0$ to be negative is

$$\left[2 - 2(1-\theta_1) f'\left(k^\theta\right) - \theta_1\right]\frac{k^\theta}{k} - (1 - \theta_0\theta_1) \leq 0$$

Using (57), this condition becomes

$$-\left[f'\left(k^\theta\right) - 1\right]^2 (1-\theta_1)\frac{k^\theta}{k} - \theta_0\theta_1 \leq 0$$

which always holds.

From (57) $dk^{\theta}/d\theta_1$ is

$$\begin{aligned} & \frac{\gamma}{\beta}\left\{\left[2-2\left(1-\theta_1\right)f'(k^{\theta})-\theta_1\right]f''\left(k^{\theta}\right)k^{\theta}+\left[f'\left(k^{\theta}\right)-1\right]\left[1-\left(1-\theta_1\right)f'(k^{\theta})\right]\right\}\frac{dk^{\theta}}{d\theta_1} \\ = \; & f''(k^{\theta})\left(1-\theta\right)\frac{dk^{\theta}}{d\theta_1}-\theta_0 f'(k^{\theta})-\frac{\gamma}{\beta}k^{\theta}\left[f'\left(k^{\theta}\right)-1\right]f'\left(k^{\theta}\right) \end{aligned}$$

Hence the condition for $\frac{dk^{\theta}}{d\theta_1}<0$ is the same as for $\frac{dk^{\theta}}{d\theta_0}<0$ and therefore we conclude that $\frac{dk^{\theta}}{d\theta_1}<0$.

Proof of Proposition 2

From (47), the Lagrange multiplier λ^{θ} is given by

$$\frac{\lambda^{\theta}}{\phi} = i^{\theta}\left(1-\theta_0\right)u'\left(q^{\theta}\right)+\theta_0 u'\left(q^{\delta}\right)\left[\left(1-\theta_1\right)\left(1+i^{\theta}\right)-1\right]$$

Using (10), (19), (39) and (51)-(54) and setting $u=\ln$ yields

$$\frac{\lambda^{\theta}}{\phi} = \frac{1}{kf'(k)}\left[f'(k)-1-\frac{\theta_0\theta_1}{1-\theta_1}\right]$$

which implies that $\lambda^{\theta}>0$ for $\theta_0=0$. Since from (54) k is independent of θ_0, λ^{θ}/ϕ is decreasing in θ_0 and we deduce that there is a value $\bar{\theta}_0^F$ such that for $\theta_0 \geq \bar{\theta}_0^F$ the multiplier λ^{θ} equals zero.

From (10), (19), (39) and (51)-(54), k, i, q and q^{δ} are unique in the equilibrium with $\lambda^{\theta}>0$. Finally, with a procedure similar to the one in the proof of Proposition 1 it can be shown that the equilibrium with $\lambda^{\theta}=0$ is unique.

Proof of Proposition 3

From the proof of Proposition 1 we deduce that $\bar{\theta}_0^I=\left[f'(k)-1\right]\left(\frac{1}{\theta_1/2}-1\right)$ with $k=\beta/\gamma$ from (24). Similarly, from the proof of Proposition 2 we deduce that $\bar{\theta}_0^F=\left[f'\left(k\right)-1\right]\left(\frac{1}{\theta_1}-1\right)$ where k is also equal to β/γ from (54). It is then straightforward that, for a given θ_1, $\bar{\theta}_0^F<\bar{\theta}_0^I$ with $\bar{\theta}_0^I=\frac{2-\theta_1}{1-\theta_1}\bar{\theta}_0^F$. From the expressions for $\bar{\theta}_0^I$ and $\bar{\theta}_0^F$ it follows that both are decreasing in θ_1. Since $\bar{\theta}_0^I$ is decreasing in θ_1, its minimum value occurs for $\theta_1=1$ which gives $\bar{\theta}_0^I=f'(k)-1$. Therefore the condition for $\bar{\theta}_0^I\geq 1$ when $\theta_1=1$ is $f'(k)>2$. It follows that the equilibrium with $\lambda=0$ fails to exist if $f'(k)>2$. However, when $\theta_1\rightarrow 1$, $\bar{\theta}_0^F\rightarrow 0$ regardless of the value of $f'\left(k\right)$. It follows that for any $k=\beta/\gamma$ there are combinations of θ_0 and θ_1 such that the equilibrium displays $\lambda^{\theta}=0$.

Proof of Proposition 4

With fragmented banking, welfare is

$$\begin{aligned}\mathcal{W} &= u(q) - q + (1-\theta_0)\left[u\left(q^\theta\right) - q^\theta\right] + \theta_0\left[u\left(q^\delta\right) - q^\delta\right] \\ &+ f(k) + (1-\theta_0\theta_1) f\left(k^\theta\right) - k - k^\theta \end{aligned} \tag{71}$$

Therefore, the effect of θ_0 on welfare is given by

$$\begin{aligned}\frac{d\mathcal{W}}{d\theta_0} &= \left[u'(q) - 1\right]\frac{dq}{d\theta_0} + (1-\theta_0)\left[u'\left(q^\theta\right) - 1\right]\frac{dq^\theta}{d\theta_0} + \theta_0\left[u'\left(q^\delta\right) - 1\right]\frac{dq^\delta}{d\theta_0} \\ &+ \left[f'(k) - 1\right]\frac{dk}{d\theta_0} + \left[(1-\theta_0\theta_1) f'\left(k^\theta\right) - 1\right]\frac{dk^\theta}{d\theta_0} \\ &- \left[u\left(q^\theta\right) - q^\theta\right] + u\left(q^\delta\right) - q^\delta - \theta_1 f\left(k^\theta\right)\end{aligned}$$

The equilibrium with $\lambda^\theta > 0$ exists for $\theta_0 < \bar{\theta}_0^F$ when CBDC is available and for $\theta_0 < 1$ when CBDC is not available. From the definition of the equilibrium with $\lambda^\theta > 0$ in fragmented banking, the equilibrium variables are constant in θ_0. Hence from (71) the effect of an increase in θ_0 on welfare in the absence of a CBDC is

$$\frac{d\mathcal{W}^{\mathcal{N}}}{d\theta_0} = -\left[u(q) - q\right] + u\left(q^\delta\right) - q^\delta - \theta_1 f(k) < 0 \tag{72}$$

where the subscript "N" indicates "no CBDC" (this derivative is the same for all θ_0).

The equilibrium with $\lambda^\theta = 0$ exists for $\theta_0 \geq \bar{\theta}_0^F$ when CBDC is available.

From (54) and (19), $dk/d\theta_0 = dq/d\theta_0 = 0$ and, from the Proof of Lemma 3, $dk^\theta/d\theta_0 < 0$. Therefore (71) yields

$$\begin{aligned}\frac{d\mathcal{W}^{\mathcal{C}}}{d\theta_0} &= (1-\theta_0)\left[u'\left(q^\theta\right) - 1\right]\frac{dq^\theta}{d\theta_0} + \theta_0\left[u'\left(q^\delta\right) - 1\right]\frac{dq^\delta}{d\theta_0} + \left[(1-\theta_0\theta_1) f'\left(k^\theta\right) - 1\right]\frac{dk^\theta}{d\theta_0} \\ &- \left[u\left(q^\theta\right) - q^\theta\right] + u\left(q^\delta\right) - q^\delta - \theta_1 f(k)\end{aligned} \tag{73}$$

where the subscript "C" indicates "with CBDC". From (49) and (50) we further obtain

$$\frac{dq^\delta}{d\theta_0} = \frac{dk^\theta}{d\theta_0}\left[(1-\theta_1) f'\left(k^\theta\right) - 1 + k^\theta (1-\theta_1) f''\left(k^\theta\right)\right]$$

and

$$\frac{dq^\theta}{d\theta_0} = \left[f'\left(k^\theta\right) + k^\theta f''\left(k^\theta\right) - 1\right]\frac{dk^\theta}{d\theta_0}$$

Since $dk^\theta/d\theta_0 < 0$, if $f'\left(k^\theta\right) + k^\theta f''\left(k^\theta\right) < 1$ then $dq^\theta/d\theta_0 > 0$ which also implies that $(1-\theta_1)\left[f'\left(k^\theta\right) + k^\theta f''\left(k^\theta\right)\right] < 1$ and therefore $dq^\delta/d\theta_0 > 0$. In this case q^θ becomes higher than q due to the increase in the interest rate in the risky country, that bears a risk premium.

Using the derivatives of q^{δ} and q^{θ} with respect to θ_0 and (56), (73) becomes

$$\begin{aligned}\frac{d\mathcal{W}^{\mathcal{C}}}{d\theta_0} &= \left\{(1-\theta_0)\left[u'\left(q^{\theta}\right)-1\right]+\theta_0\left[u'\left(q^{\delta}\right)-1\right](1-\theta_1)\right\}k^{\theta}f''\left(k^{\theta}\right)\frac{dk^{\theta}}{d\theta_0} \\ &- \left[u\left(q^{\theta}\right)-q^{\theta}\right]+u\left(q^{\delta}\right)-q^{\delta}-\theta_1 f\left(k^{\theta}\right) \end{aligned} \tag{74}$$

Comparing (72) with (74), it is straightforward that $d\mathcal{W}^{\mathcal{C}}/d\theta_0 > d\mathcal{W}^{\mathcal{N}}/d\theta_0$ for $\theta_0 \geq \bar{\theta}_0^F$. This implies that $\mathcal{W}^C > \mathcal{W}^N$ for all $\theta_0 \geq \bar{\theta}_0^F$.

Proof of Proposition 5

Let $\gamma > 1$. From (62), it is immediate that $\theta_0^{DI} > 0$. Define the function $y(x) = \frac{\ln(1-x)}{x}$ which is continuous and strictly decreasing on $(0,1)$ with $\lim_{x\to 1^-} y = -\infty$. Differentiating y gives

$$y' = -\frac{1}{x^2}\left(\frac{x}{1-x}+\ln(1-x)\right) \equiv -\frac{1}{x^2}\times G(x) \tag{75}$$

Now, one can easily check that $G(0)=0$ and that $G'(x) = \frac{x}{(1-x)^2} > 0$ so that $G(x) > 0$ $\forall x \in (0,1)$. Hence, $y' < 0$, proving that y is decreasing. The limit when $x \to 0^+$ comes from $\ln(1-x) \approx -x$ when $x \approx 0$; the limit for $x \to 1^-$ is trivial. Since $\ln(1-x) \approx -x$ when $x \approx 0$, we also have $\lim_{x\to 0^+} y = -1$. This proves that $\frac{\partial \theta_0^{DI}}{\partial \theta_1} < 0$, $\lim_{\theta_1 \to 0^+} \theta_0^{DI} = \frac{2(\gamma-1)kf'(k)}{(1+\chi)}\frac{1}{1-q}$ and $\lim_{\theta_1\to 1^-} \theta_0^{DI} = 0$. It follows from the condition for $\mathcal{W}^{DI} > \mathcal{W}^{\mathcal{N}}$ and the definition of θ_0^{DI} in (62), that full insurance dominates no deposit insurance for $\theta_0 > \theta_0^{DI}$.

Proof of Proposition 6

To compare θ_0^{DI} and $\bar{\theta}_0^F$, recall that

$$\bar{\theta}_0^F = \frac{1-\theta_1}{\theta_1}\left[f'(k)-1\right]$$

Given (62), $\bar{\theta}_0^F < \theta_0^{DI}$ is equivalent to

$$-\left(\frac{\ln(1-\theta_1)}{\theta_1}+q\right)\frac{1-\theta_1}{\theta_1} < \frac{2(\gamma-1)kf'(k)}{(1+\chi)}\frac{1}{f'(k)-1}$$

Let

$$z(x) = -\left(\frac{\ln(1-x)}{x}+q\right)\frac{1-x}{x}$$

It can be shown that the function $z(x)$ is continuous and strictly decreasing on $(0,1)$, with $\lim_{x\to 0^+} z = +\infty$ and $\lim_{x\to 1^-} z = 0$. The limits as $x \to 0^+$ follows from that for y in the proof of Proposition 5, while the limit as $x \to 1^-$ follows from $\lim_{x\to 0^+} x\ln(x) = 0$. To show monotonicity, we need to show that $z' < 0$. Rewrite $z(x)$ as

$$z(\theta_1) = -(y(x)+q)\frac{1-x}{x}$$

Hence, z' is

$$z' \;=\; -\left(y'(x)\frac{1-x}{x} + (y(x)+q)\frac{-1}{x^2}\right) \tag{76}$$

Observe that, from the first part of (75),

$$y' = -\frac{1}{x}\left(\frac{1}{1-x} + y(x)\right)$$

Inserting into (76) yields

$$z' \;=\; \frac{1}{x^2}\left((2-x)\,y(x) + 1 + q\right)$$

where

$$[(2-x)\,y]' \;=\; -\frac{1}{x^2}\left(2\ln(1-x) + x\frac{2-x}{1-x}\right) \equiv -\frac{1}{x^2}\times H(x) \tag{77}$$

with $H(0) = 0$ and

$$H' \;=\; \frac{x^2}{(1-x)^2} > 0$$

Thus $H(x) \geq 0$, implying that $(2-x)\,y$ is decreasing, and $(2-x)\,y < \lim_{x\to 0^+}(2-x)\,y = -2$. Using (77), it follows that $z' < 0$. This proves that there is $\hat{\theta}_1$ such that $\bar{\theta}_0^F > \theta_0^{DI}$ for $\theta_1 < \hat{\theta}_1$ and $\bar{\theta}_0^F < \theta_0^{DI}$ for $\theta_1 > \hat{\theta}_1$. Hence, if $\theta_1 < \hat{\theta}_1$, we have $\bar{\theta}_0^F > \theta_0^{DI}$ and therefore $\mathcal{W}^{DI} > \mathcal{W}^C$ for $\theta_0 \in \left(\theta_0^{DI}, \bar{\theta}_0^F\right)$ since for that range of values of θ_0 introducing a CBDC leaves the the equilibrium and therefore the welfare level unaffected while implementing a deposit insurance improves welfare. If instead $\theta_1 > \hat{\theta}_1$ we have $\bar{\theta}_0^F < \theta_0^{DI}$, and therefore $\mathcal{W}^C > \mathcal{W}^{DI}$ for $\theta_0 \in \left(\bar{\theta}_0^F, \theta_0^{DI}\right)$ since for those values of θ_0 implementing a CBDC improves welfare while setting a deposit insurance is welfare worsening.

Finally, for the expression for welfare with deposit insurance, note that total utility in the settlement market is the sum of the utility of sellers, suppliers, entrepreneurs and buyers (since banks make zero profits in equilibrium), each with measure 2 when considering the two countries. For each supplier, the utility is equal to k whether a macroeconomic shock occurs or not. In the absence of a macroeconomic shock, each entrepreneur obtains $f(k) - \phi\ell(1+i)$, each seller $\phi d(1+i)$ and each buyer $-\phi m_{+1} + \phi T - \phi\tau + \phi\tau_{-1}$ which simplifies to $-\phi m - (\gamma-1)\,\phi\tau_{-1} = -\phi m - (\gamma-1)\,\tilde{\tau}\phi\ell^\theta\left(1+i^\theta\right)$ with $\phi m = \phi d = \phi\ell = \phi p_k k = k$, $\phi\ell^\theta\left(1+i^\theta\right) = k^\theta f'\left(k^\theta\right)$ and $k^\theta = k$. If a shock occurs, $2-\theta_1$ entrepreneurs enjoy their production surplus. Every seller in the risky country obtains, for his deposits, $\phi\delta^{DI}d^\theta = [1-\theta_1+(1+\chi)\,\tilde{\tau}]\left(1+i^\theta\right)\phi d^\theta$. In the safe country the utility for a buyer is $-\phi m - \phi\tau + (1-\chi)\,\phi\tau_{-1} = -k - [\gamma-(1-\chi)]\,\tilde{\tau}\phi\ell^\theta\left(1+i^\theta\right)$, while in the risky country it is $-\phi m - \phi\tau = -k - \gamma\tilde{\tau}\phi\ell^\theta\left(1+i^\theta\right)$. Therefore, taking into account the equilibrium values for $u = \ln$, in expected terms the sum of all utilities in the SM is $(2-\theta) f(k) - 2(\gamma - 1)\tilde{\tau} k f'(k)$.

Buyers' program under fragmented banking without CBDC

Without CBDC, money cannot be used to acquire goods. Since in the bad state buyers spend all their deposits in the risky country, their program is

$$\max_{d^\theta, q_d^\theta, q^\delta} \quad (1-\theta_0)\left[u\left(q^\theta\right) + W^\theta\left(m - d^\theta, d^\theta - p_d^\theta q_d^\theta\right)\right] + \theta_0 u\left(q^\delta\right)$$

subject to

$$\begin{aligned} d^\theta &\leq m \\ q^\theta &\leq \frac{d^\theta}{p_d^\theta} \\ q^\delta &= \frac{d^\theta}{p_d^\delta} \end{aligned}$$

The buyer's first-order conditions on q^θ and d^θ are

$$(1-\theta_0)\left[u'\left(q^\theta\right) - \phi p_d^\theta\left(1 + i_d^\theta\right)\right] = \mu_d^\theta$$

and

$$\frac{\mu_d^\theta}{p_d^\theta} + (1-\theta_0)\phi i_d^\theta + \theta_0\left[u'\left(q^\delta\right)\frac{1}{p_d^\delta} - \phi\right] = \lambda^\theta$$

where λ^θ and μ_d are the Lagrange multipliers on the first and second constraints. Using (37), and the sellers' indifference conditions (41) and (42), we have

$$(1-\theta_0)\left[u'\left(q^\theta\right) - 1\right] = \mu_d^\theta$$

and

$$(1-\theta_0)\left[u'\left(q^\theta\right) - 1\right]\left(1 + i_d^\theta\right) + (1-\theta_0) i_d^\theta + \theta_0\left[u'\left(q^\delta\right)(1-\theta_1)\left(1 + i^\theta\right) - 1\right] = \frac{\lambda^\theta}{\phi} \quad (78)$$

The first-order condition on money holdings is

$$\frac{\gamma}{\beta} = \frac{\lambda^\theta}{\phi} + 1 \quad (79)$$

Therefore $\lambda^\theta > 0$ always holds in equilibrium for $\gamma > \beta$.

Model with money creation

In this extension we assume that a fraction $\alpha \leq 1$ of suppliers require the use of central-bank money, either because they are unbanked or because they withdraw their deposits upon being paid in the capital market (for example, because they learn that they will not have access to banks in the following settlement market). This assumption implies that banks must keep at least a fraction α of the issued deposits as reserves (central-bank money, either physical or digital). In the baseline model, we had $\alpha = 1$. We present below the different equiliibria for a generic $\alpha \leq 1$. We focus on the case $u'(q), u'\left(q^{\theta}\right) > 1$ (it is straightforward to show the other cases as well).

With integrated banking, the definition of the equilibrium with $\lambda > 0$ and with $\lambda = 0$ are as follows.

Definition 8 *An equilibrium with $\lambda > 0$ is $\{k, i, q, q^0\}$ that solve, recursively,*

$$
\begin{aligned}
\frac{\beta}{\gamma} &= k\left[1 - f'(k)(1-\alpha)(1-\theta/2)\right] \\
1+i &= \frac{\alpha}{\frac{1}{f'(k)} - (1-\alpha)(1-\theta/2)} \\
q &= \alpha f'(k)\, k \\
q^0 &= \alpha f'(k)\, k\,(1-\theta_1/2)
\end{aligned}
$$

Definition 9 *An equilibrium with $\lambda = 0$ is $\{k, i, q, q^0\}$ that solve, recursively,*

$$
\begin{aligned}
\frac{\gamma}{\beta} k\left[1 - f'(k)\left[1-(1-\alpha)\theta/2\right]\right] &= \frac{1 - f'(k)(1-\theta/2)}{1 - f'(k)\left[1-(1-\alpha)\theta/2 - \alpha\theta_1/2\right]} \\
1+i &= \frac{\alpha}{\frac{1}{f'(k)} - (1-\alpha)(1-\theta/2)} \\
\frac{\gamma}{\beta} q &= \frac{(1-\theta_0)(\theta_1/2)(1+i)}{1-(1-\theta_1/2)(1+i)} \\
q^0 &= q - \alpha k f'(k)\,\theta_1/2
\end{aligned}
$$

It can be shown that in the equilibrium with $\lambda > 0$ with integrated banking, if $\alpha < 1$, k, q, and q^0 are decreasing both in θ_0 and θ_1. If $\alpha = 1$, k and q are constant in θ_0 and θ_1 and, q^0 is constant in θ_0 and decreasing in θ_1. The proposition and the proof of existence are analogous to the one for $\alpha = 1$.

With fragmented banking the definition of the equilibrium with $\lambda^{\theta} > 0$ and with $\lambda^{\theta} = 0$ are as follows.

Definition 10 *An equilibrium with* $\lambda^{\theta} > 0$ *is* $\left\{q, q^{\theta}, q^{0}, i, i^{\theta}, k, k^{\theta}\right\}$ *that solve, recursively,*

$$\begin{aligned}
\frac{\beta}{\gamma} &= k\left[1 - f'(k)(1-\alpha)\right] \\
\frac{\beta}{\gamma} &= k^{\theta}\left[1 - f'\left(k^{\theta}\right)(1-\alpha)(1-\theta)\right] \\
f'(k) &= \frac{1}{1-\alpha+\alpha(1+i)^{-1}} \\
\frac{f'(k)\,k}{1+i} &= \frac{f'\left(k^{\theta}\right)k^{\theta}}{1+i^{\theta}} \\
q &= \alpha f'(k)\,k \\
q^{\theta} &= \alpha f'\left(k^{\theta}\right)k^{\theta} \\
q^{0} &= \alpha f'\left(k^{\theta}\right)k^{\theta}(1-\theta_1)
\end{aligned}$$

Definition 11 *An equilibrium with* $\lambda^{\theta} = 0$ *is* $\left\{q, q^{\theta}, q^{0}, i, i^{\theta}, k, k^{\theta}\right\}$ *that solve, recursively,*

$$\begin{aligned}
\frac{\beta}{\gamma} &= k\left[1 - f'(k)(1-\alpha)\right] \\
\frac{\beta}{\gamma} &= k^{\theta}\frac{1 + f'\left(k^{\theta}\right)\left[\alpha\theta_1 - 1 + \theta(1-\alpha)\right]}{1 + \frac{f'\left(k^{\theta}\right)\alpha\theta}{1 - f'\left(k^{\theta}\right)\left[(1-\alpha)(1-\theta)+\alpha\right]}} \\
f'(k) &= \frac{1}{1-\alpha+\alpha(1+i)^{-1}} \\
f'\left(k^{\theta}\right) &= \frac{1}{(1-\alpha)(1-\theta)+\alpha\left(1+i^{\theta}\right)^{-1}} \\
q &= \alpha f'(k)\,k \\
q^{0} &= \frac{\alpha f'(k)\,k}{1+i} + \frac{\alpha f'\left(k^{\theta}\right)k^{\theta}\left[(1-\theta_1)\left(1+i^{\theta}\right)-1\right]}{1+i^{\theta}} \\
q^{\theta} &= q^{0} + \alpha f'\left(k^{\theta}\right)k^{\theta}\theta_1
\end{aligned}$$

Comparing the equilibrium with $\lambda > 0$ under integrated banking and the equilibrium with $\lambda^{\theta} > 0$ under fragmented banking, we deduce that the level of invested capital in the safe country is higher with fragmented banking. Investing in banks in the safe country under fragmented banking is indeed safer than investing in banks in the integrated setting. Because of this, with highly risk averse agents, money demand with fragmented banking may be higher, which favors investment in the risky country as well.

It can be shown that in the equilibrium with $\lambda^{\theta} > 0$ with fragmented banking, k is constant in θ_0 while if $\alpha < 1$ k^{θ} is decreasing in θ_0 and if $\alpha = 1$ k^{θ} is constant in θ_0. The proposition and the proof of existence are analogous to the one for $\alpha = 1$.

With fragmented banking, the level of invested capital is different across countries. This feature makes this equilibrium different from the one that prevails with an integrated banking sector where all banks hold identical portfolios, regardless of their location. In this case, there is only one type of deposit and consequently a unique price of capital in terms of deposits. With fragmented banking, only if $\alpha = 1$ and deposits are fully backed by reserves (central-bank money), all capital sells at the same price. In this case, there is a unique level of invested capital across countries both under integrated and fragmented banking in the equilibria with binding deposit constraints. However, even with $\alpha = 1$, while with integrated banking there is still one unique level of invested in both countries, this is not the case under fragmented banking, once we consider in the equilibria with slack deposit constraints. The reason is that, with fragmentation in the banking sector, the volume of lending by banks in the risky country is smaller than the one in the safe country, as depositors in the risky country deposit only a fraction of their money holdings and a partial disintermediation occurs. Thus, a difference emerges between banks in the safe and the risky country when they are fragmented, since only depositors in the safe country deposit all their money holdings, whereas such a difference does not arise under integrated banking.

Finally, the comparison between $\bar{\theta}_0^F$ and $\bar{\theta}_0^I$ is as in the baseline model.