Optimal Stabilization Policy with Flexible Prices*

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Abstract

We construct a dynamic stochastic general equilibrium model to study optimal monetary stabilization policy. Unlike existing New Keynesian models prices are fully flexible and money is essential for trade. Our main result is that if the central bank commits to a long-run price path, it can successfully stabilize short-run aggregate shocks to the economy and improve welfare. The optimal policy involves smoothing nominal interest rates which effectively smooths consumption across states. If it cannot commit to such a path, any stabilization attempts are ineffective.

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1 Introduction

After a long period of inactivity, the last few years have seen a tremendous resurgence of research focusing on the question of how to conduct optimal monetary policy. Nearly all of this work has come from the New Keynesian literature which, in the tradition of real business cycle models, constructs dynamic stochastic general equilibrium models to study optimal stabilization policy. What separates New Keynesian models from real business cycle models is their reliance on nominal rigidities, such as price or wage stickiness, that allows monetary policy to have real effects. Without these nominal rigidities there is no role for stabilization policy since money is neutral. From this research one is tempted to conclude that price stickiness is necessary to generate a role for stabilization policy. In this paper we show that this is not the case – there is a welfare improving role for stabilization policy even if prices are fully flexible.

In our model the critical element for effective stabilization policy is the central bank's commitment to a price path. Commitment to a price path allows the central bank to control inflation expectations so that monetary policy has real effects even though prices are fully flexible. What is interesting about our result is that it is closely related to a key policy recommendation coming out of the New Keynesian models which is that "good" monetary policy requires commitment to a price or inflation target in order to control inflation expectations.[1] In the New Keynesian models controlling inflation expectations allows the central bank to implement a more effective stabilization response to aggregate shocks. Thus, our model extents this policy recommendation to environments in which prices are fully flexible. What is new about our result is that, in contrast to New Keynesian models, stabilization policy is completely

[1] See, for example, Woodford 2003 Chapters 1 and 7. Also see Clarida, Gali and Gertler (1999) p. 1663.

neutral without commitment. Hence, controlling inflation expectations is even more important when prices are flexible.

To show the importance of commitment for stabilization policy, we construct a dynamic stochastic general equilibrium model where money is essential for trade and prices are fully flexible.[2] There are aggregate shocks to preferences and technology. The existence of a credit sector generates a nominal interest rate that the monetary authority manipulates in its attempt to stabilize these shocks. Policy is optimal since the monetary authority maximizes the expected lifetime utility of the representative agent subject to the allocation being a competitive equilibrium.

Our main result is that if the central bank commits to a long-run price path, it can successfully stabilize short-run aggregate shocks to the economy and improve welfare. The optimal policy involves smoothing nominal interest rates which effectively smooths consumption across states. If it cannot commit to such a path, any stabilization attempts are ineffective. With commitment stabilization policy works through a liquidity effect. By injecting money into the economy the central bank lowers nominal interest rates. Without commitment these injections simply raise price expectations and the nominal interest rate, as predicted by the Fisher equation, without affecting the real allocation.

The paper proceeds as follows. In Section 2 we describe the environment. In Section 3 agents' optimization problems are presented and in Section 4 we derive the optimal monetary policy. Section 5 contains discussion of the results and Section 6 concludes.

[2] By essential we mean that the use of money expands the set of allocations (Kocherlakota (1998) and Wallace (2001)).

2 The Environment

The basic environment is that of Berentsen et al. (2004) which builds on Lagos and Wright (2005). We use the Lagos-Wright framework because it provides a microfoundation for money demand and it allows us to introduce heterogenous preferences for consumption and production while keeping the distribution of money balances analytical tractable.[3] Time is discrete and in each period there are two perfectly competitive markets that open sequentially.[4] There is a $[0, 1]$ continuum of infinitely-lived agents and one perishable good produced and consumed by all agents.

At the beginning of the first market agents receive a preference shock such that they either consume, produce or neither. With probability n an agent consumes, with probability s he produces and with probability $1 - n - s$ he is inactive in the goods market. We refer to consumers as buyers and producers as sellers. Buyers get utility $\varepsilon u(q)$ from $q > 0$ consumption in the first market, where ε is a preference parameter and $u'(q) > 0$, $u''(q) < 0$, $u'(0) = +\infty$ and $u'(\infty) = 0$. Furthermore, we impose that the elasticity of utility $e(q) = \frac{qu'(q)}{u(q)}$ is bounded. Producers incur utility cost $c(q)/\alpha$ from producing q units of output where α is a measure of productivity. We assume that $c'(q) > 0$, $c''(q) \geq 0$ and $c'(0) = 0$.

Following Lagos and Wright (2005) we assume that in the second market all agents consume and produce, getting utility $U(x)$ from x consumption, with $U'(x) > 0$, $U'(0) = \infty$, $U'(+\infty) = 0$ and $U''(x) \leq 0$.[5] Agents can produce one

[3] An alternative framework would be Shi (1997) which we could ammend with preference and technology shocks to generate the same results.

[4] Competitive pricing in the Lagos-Wright framework is a feature in Rocheteau and Wright (2005) and Berentsen et al. (2005).

[5] The difference in preferences over the good sold in the last market allows us to impose technical conditions such that the distribution of money holdings is degenerate at the beginning of a period.

unit of the consumption good with one unit of labor which generates one unit of disutility. The discount factor across dates is $\beta \in (0, 1)$.

To motivate a role for fiat money, we assume that all goods trades are anonymous. In particular, trading histories of agents are private information. Consequently, sellers require immediate compensation so buyers pay with money. There is also no public communication of individual trading outcomes (public memory), which eliminates the use of trigger strategies to support gift-giving equilibria.

2.1 Credit

At the beginning of a period, after the idiosyncratic shocks are realized, sellers and inactive agents hold idle money balances while buyers may have a desire for more money. This inefficiency creates a role for financial intermediation. As in Berentsen et al. (2004) we assume that a perfectly competitive banking sector creates this market. Banks accept nominal deposits and pay the nominal interest rate i_d and make nominal loans at nominal rate i. Since the banking sector is perfectly competitive, banks take these rates as given. There are no operating costs so zero profits imply $i_d = i$. Banks have a record-keeping technology over financial transactions.

In the first market the banking sector opens and agents borrow and deposit after observing the shocks. Then, they trade. In the second market all financial claims are settled. This essentially means that loans and deposits cannot be rolled over. Consequently, all financial contracts are one-period contracts. In all models with credit default is a serious issue. However, to focus on optimal stabilization, we simplify the analysis by assuming that default is not feasible.[6]

[6] In Berentsen et al. (2004) we derive the equilibrium when the only punishment for default is exclusion from the banking system in all future periods. Alternatively, we could assume that agents require a particular type of 'tool' to be able to consume in market

2.2 Aggregate shocks

To study the optimal response to aggregate shocks we assume that n, s, α and ε are stochastic. The random variable n has support $[\underline{n}, \overline{n}] \in (0, 1/2]$, s has support $[\underline{s}, \overline{s}] \in (0, 1/2]$, α has support $[\underline{\alpha}, \overline{\alpha}]$, $0 < \underline{\alpha} < \overline{\alpha} < \infty$, and ε has support $[\underline{\varepsilon}, \overline{\varepsilon}]$, $0 < \underline{\varepsilon} < \overline{\varepsilon} < \infty$. Let $\omega = (n, s, \alpha, \varepsilon) \in \Omega$ be the aggregate state in market 1, where $\Omega = [\underline{n}, \overline{n}] \times [\underline{s}, \overline{s}] \times [\underline{\alpha}, \overline{\alpha}] \times [\underline{\varepsilon}, \overline{\varepsilon}]$ is a closed and compact subset on $\boldsymbol{R}_{+}^{4}$. The shocks are serially uncorrelated. Let $f(\omega)$ denote the density function of ω.

Shocks to n and ε are aggregate demand shocks, while shocks to s and α are aggregate supply shocks. We call shocks to ε and α intensive margin shocks since they change the desired consumption of each buyer and the productivity of each seller, respectively, without affecting the number of buyers or sellers. In contrast, shocks to n and s affect the number of buyers and sellers.

2.3 Monetary Policy

Monetary policy has a long and short-run component. The long-run component focuses on the trend inflation rate. The short-run component is concerned with the stabilization response to aggregate shocks.

We assume a central bank exists that controls the supply of fiat currency. We denote the gross growth rate of the money supply by $\gamma = M_t/M_{t-1}$ where M_t denotes the per capita money stock in market 2 in period t. The central bank implements its long-term inflation goal by providing deterministic lump-sum injections of money, τM_{t-1}, at the beginning of period 1. These transfers are given to the private agents rather than to the banks.[7] The net change in the

2. This tool can be used as collateral against loans in market 1 and for sufficiently high discount factors repayment occurs with probability one.

[7]Alternatively, the transfers could be given to the banks for intermediation. But for technical reasons related to the zero profit condition of banks, the analysis is much easier

aggregate money stock is the given by $\tau M_{t-1} = (\gamma - 1)M_{t-1}$. If $\gamma > 1$, agents receive lump-sum transfers of money. For $\gamma < 1$, the central bank must be able to extract money via lump-sum taxes from the economy. We study monetary equilibria for two cases. In one case we assume the central bank has the ability to levy lump-sum taxes in the form of currency and in the other case it cannot implying $\gamma \geq 1$ for all t. For notational ease variables corresponding to the next period are indexed by $+1$, and variables corresponding to the previous period are indexed by -1.

The central bank implements its short-term stabilization policy through state contingent changes in the stock of money. Let $\tau_1(\omega) M_{-1}$ and $\tau_2(\omega) M_{-1}$ denote the state contingent cash injections in market 1 and 2 received by private agents. We assume that $\tau_1(\omega) + \tau_2(\omega) = 0$. In short, any injections in market 1 are undone in market 2. This effectively means that the long-term inflation rate is still deterministic since τM_{-1} is not state dependent. Consequently, changes in $\tau_1(\omega)$ allows us to affect the money stock for stabilization purposes without affecting the long-term inflation rate.[8] With $\tau_1(\omega) + \tau_2(\omega) = 0$ we are implicitly assuming the central bank is committed to a given path of the money stock in market 2. As we show later, this commitment allows the central bank to control price expectations in market 2 which is critical for successful stabilization policy. Without loss of generality we can assume that $\tau_1(\omega) \geq 0$ in all states.

The state contingent injections of cash should be viewed as a type of repurchase agreement – the central bank 'sells' money in market one under the agreement that it is being repurchased in market 2. Alternatively, $\tau_1(\omega) M_{-1}$ can be thought of as a zero interest discount loan to households that is repaid in the night market. To ensure that the contracts are carried out we assume

by assuming that they are given to private agents directly.

[8] Lucas (1990) employs a similar process for the money supply so that changes in nominal interest rates result purely from liquidity effects and not changes in expected inflation.

the central bank has the same recordkeeping technology and enforcement technology as private banks. Thus, the only difference between the central bank and any private bank is the ability of the central bank to print fiat currency.

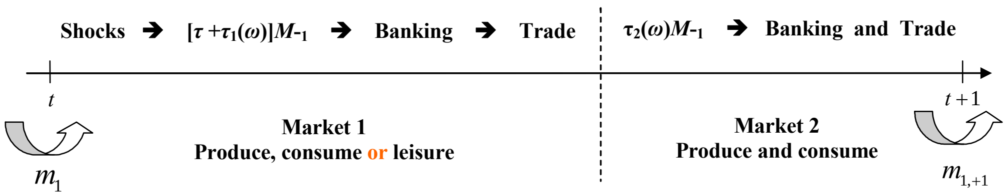

Figure 1: Sequence of events.

The precise sequence of action after the shocks are observed is as follows. The monetary injection τM_{-1} occurs and the central bank offers up to $\tau_1(\omega) M_{-1}$ units of cash per capita to agents at no cost. Sellers and inactive agents deposit their idle cash and buyers borrow money from the banking sector. Agents then move on to the goods market and trade. In the second market the goods market and banking sector open where all financial claims are settled. The central bank takes out $\tau_2(\omega) M_{-1} = -\tau_1(\omega) M_{-1}$ units of money.

3 Agents' Choices and Value functions

In period t, let P denote the nominal price of goods in the second market. It then follows that $\phi = 1/P$ is the real price of money. We study equilibria where end-of-period real money balances are time and state invariant

$$\phi M = \phi_{-1} M_{-1} \equiv z, \qquad \omega \in \Omega. \tag{1}$$

We refer to it as a stationary equilibrium. This implies that ϕ is not state dependent and so $\phi_{-1}/\phi = P/P_{-1} = M/M_{-1} = \gamma$. This effectively means that the central bank commits to a price path $P = \gamma P_{-1}$ in market 2.

Consider a stationary equilibrium. Let $V(m_1)$ denote the expected value from trading in market 1 with m_1 money balances. Let $W(m_2, l, d)$ denote the expected value from entering the second market with m_2 units of money, l loans, and d deposits when the aggregate state is ω. Note that all quantities and prices are functions of the aggregate state ω, i.e., $m_2 = m_2(\omega)$, $l = l(\omega)$, and $d = d(\omega)$. We suppress this dependence for notational simplicity. In what follows, we look at a representative period t and work backwards from the second to the first market to examine the agents' choices.

3.1 The second market

In the second market agents consume x, produce h, and adjust their money balances taking into account cash payments or receipts from the bank. Loans are repaid by borrowers and banks redeem deposits. If an agent has borrowed l units of money, then he pays $(1+i)\,l$ units of money. If he has deposited d units of money, he receives $(1+i)\,d$. The representative agent's program is

$$W(m_2, l, d) = \max_{x,h,m_{1,+1}} \left[U(x) - h + \beta V(m_{1,+1})\right] \tag{2}$$

$$\text{s.t.}\quad x + \phi m_{1,+1} = h + \phi\,(m_2 + \tau_2 M_{-1}) + \phi\,(1+i)\,d - \phi\,(1+i)\,l$$

where $m_{1,+1}$ is the money taken into period $t+1$.

Rewriting the budget constraint in terms of h and substituting into (2) yields

$$\begin{aligned} W(m_2, l, d) &= \phi\left[m_2 + \tau_2 M_{-1} - (1+i)\,l + (1+i)\,d\right] \\ &\quad + \max_{x,m_{1,+1}} \left[U(x) - x - \phi m_{1,+1} + \beta V(m_{1,+1})\right]. \end{aligned}$$

The first-order conditions are $U'(x) = 1$ and

$$-\phi_{-1} + \beta V'(m_1) = 0 \tag{3}$$

where the first-order condition for money has been lagged one period. Thus, $V'(m_1)$ is the marginal value of taking an additional unit of money into the first market open in period t. Since the marginal disutility of working is one, $-\phi_{-1}$ is the utility cost of acquiring one unit of money in the second market of period $t-1$.

The envelope conditions are

$$W_m = \phi \tag{4}$$

$$W_d = -W_l = \phi(1+i). \tag{5}$$

As in Lagos and Wright (2005) the value function is linear in wealth. The implication is that all agents enter the following period with the same amount of money.

3.2 The first market

Let q_b and q_s respectively denote the quantities consumed by a buyer and produced by a seller trading in market 1. Let p be the nominal price of goods in market 1. It is straightforward to show that buyers will never deposit funds in the bank and sellers and inactive agents will never take out loans. It is straightforward to show that it is optimal for sellers and inactive agents to deposit all their money balances if $i > 0$. If $i = 0$, they are indifferent since they earn no money. In what follows we assume that they also deposit their money when $i = 0$. Let l denote loans taken out by buyers and d deposits of sellers and inactive agents. Since the central bank offers $\tau_1(\omega) M_{-1}$ units of money at a zero interest rate, it is trivial to show that all agents acquire $\tau_1(\omega) M_{-1}$ units of cash from the central bank before going to the private banks to deposit and borrow.

An agent who has m_1 money at the opening of the first market has expected lifetime utility

$$\begin{aligned} V(m_1) = & \int_\Omega \{n[\varepsilon u(q_b) + W(m_1 + (\tau+\tau_1)M_{-1} + l - pq_b, l, 0)] \\ & +s[-c(q_s)/\alpha + W(m_1 + (\tau+\tau_1)M_{-1} - d + pq_s, 0, d)] \\ & +(1-n-s)W(m_1 + (\tau+\tau_1)M_{-1} - d, 0, d)\} f(\omega)\, d\omega \end{aligned} \tag{6}$$

where pq_b is the amount of money spent as a buyer and pq_s the money received as a seller.

A seller's problem is $\max_{q_s}[-c(q_s)/\alpha + W(pq_s, 0, d)]$. Using (4), the first-order conditions are

$$c'(q_s) = \alpha p\phi, \qquad \omega \in \Omega. \tag{7}$$

Note that sellers cannot deposit receipts of cash obtained from selling output.

If an agent is a buyer in the first market, his problem is:

$$\max_{q_b,l}[\varepsilon u(q_b) + W(m_1 + (\tau+\tau_1)M_{-1} + l - pq_b, l, 0)]$$

$$\text{s.t.} \quad pq_b \leq m_1 + (\tau+\tau_1)M_{-1} + l$$

Notice that buyers can spend more cash than what they bring into the first market since they can borrow cash to supplement their money holdings at the cost of the nominal interest rate. Using (4) the buyer's first-order conditions are

$$\varepsilon u'(q_b) - p\phi - p\lambda = 0, \qquad \omega \in \Omega, \tag{8}$$

$$-i\phi + \lambda = 0, \qquad \omega \in \Omega, \tag{9}$$

$$\lambda[m_1 + (\tau+\tau_1)M_{-1} + l - pq_b] = 0, \qquad \omega \in \Omega, \tag{10}$$

where $\lambda = \lambda(\omega)$ are the multipliers of the buyer's budget constraints.

Equations (7), (8) and (9) imply that

$$\alpha\varepsilon u'(q_b) = c'(q_s)(1+i), \qquad \omega \in \Omega \tag{11}$$

If the constraint is not binding, (9) implies $i = 0$ and so (11) reduces to $\alpha\varepsilon u'(q_b) = c'(q_s)$. Hence trades are efficient if $\lambda = 0$.[9] If the constraint is binding, then (9) implies $i > 0$ and so $\alpha\varepsilon u'(q_b) > c'(q_s)$, which means trades are inefficient. The buyer spends all of his money, $pq_b = m_1 + (\tau + \tau_1) M_{-1} + l$, and consumes $q_b = [m_1 + (\tau + \tau_1) M_{-1} + l]/p$.

We show in the proof of Lemma 1 in the appendix that the marginal value of money is given by

$$V'(m_1) = \int_{\Omega} \phi \{n\alpha\varepsilon u'(q_b)/c'(q_s) + (1-n)(1+i)\} f(\omega)\, d\omega. \tag{12}$$

Note that banks increase the marginal value of money because agents can earn interest on idle money as opposed to the non-bank case where $i = 0$.

Then, using (11) we can write $V'(m_1)$ as follows

$$V'(m_1) = \int_{\Omega} [\phi\alpha\varepsilon u'(q_b)/c'(q_s)] f(\omega)\, d\omega. \tag{13}$$

Differentiating (13) shows that the value function is concave in m_1.

3.3 Equilibrium

We now derive the symmetric monetary steady-state equilibrium. Since in a symmetric equilibrium all sellers produce the same quantity, market clearing in the good market implies

$$q(\omega) \equiv q_b(\omega) = (s/n)\, q_s(\omega), \qquad \omega \in \Omega. \tag{14}$$

Since in a symmetric equilibrium all borrowers take out the same loan, $l(\omega)$, and depositors deposit the same amount $d(\omega)$ market clearing in the credit market implies

$$l(\omega) = \frac{1-n}{n} d(\omega), \qquad \omega \in \Omega. \tag{15}$$

[9] With n buyers and s sellers, the first-best quantities are obtained by maximizing $n\varepsilon u(q_b) - (s/\alpha)\, c(q_s)$ s.t. $nq_b = sq_s$ for all $\omega \in \Omega$. The first-order conditions for q_b are $\alpha\varepsilon u'(q_b) = c'(q_s)$ for all $\omega \in \Omega$.

Since in equilibrium $m_1 = M_{-1}$ and $d(\omega) = [1 + \tau + \tau_1(\omega)] M_{-1}$ the budget constraint of the buyer has to satisfy $pq(\omega) \leq [1 + \tau + \tau_1(\omega)] M_{-1}/n$. Then from (7), (14) and (15) we get

$$(n/\alpha)\, q(\omega)\, c'[(n/s)\, q(\omega)] \leq [1 + \tau + \tau_1(\omega)]\, \phi M_{-1}. \tag{16}$$

In any monetary equilibrium (16) holds with equality in at least one state. In these states we have

$$(n/\alpha)\, q(\omega)\, c'[(n/s)\, q(\omega)] = \upsilon(\omega)\, z. \tag{17}$$

where $z = \phi M$ is the real stock of money and $\upsilon(\omega) = [1 + \tau + \tau_1(\omega)] / (1 + \tau)$. For any given state ω, $q(\omega)$ is an increasing function of z. Consequently, we have

$$\begin{aligned} q(\omega, z) &< q^*(\omega) \text{ if } \lambda(\omega) > 0 \\ &\text{and} \\ q(\omega) &= q^*(\omega) \text{ if } \lambda(\omega) = 0 \end{aligned} \tag{18}$$

where the efficient quantity $q^*(\omega)$ solves $\alpha\varepsilon u'[q^*(\omega)] = c'[(n/s)\, q^*(\omega)]$.

Use (3) to eliminate $V'(m_1)$ and (14) to eliminate q_s from (13). Then, multiply the resulting expression by M_{-1} to get

$$\frac{\gamma - \beta}{\beta} = \int_{\Omega} \left\{ \frac{\alpha\varepsilon u'[q(\omega, z)]}{c'[(n/s)\, q(\omega, z)]} - 1 \right\} f(\omega)\, d\omega. \tag{19}$$

Definition 1 *A symmetric monetary steady-state equilibrium is a z that satisfies (19).*

We define the equilibrium as the value of z that solves (19). The reason is that once the equilibrium stock of money is determined all other endogenous variables can be derived recursively. For example, for a given z equations (17) and (18) yield $q(\omega)$ for all $\omega \in \Omega$.

4 Optimal stabilization

We now derive the optimal stabilization policy. We assume that the central bank's objective is to maximize the welfare of the representative agent. It does so by choosing the quantities consumed and produced in each state subject to the constraint that the chosen quantities satisfy the conditions of a competitive equilibrium. The policy is implemented by choosing state contingent injections $\tau_1(\omega)$ and $\tau_2(\omega)$ accordingly.

It is straightforward to show that the expected lifetime utility of the representative agent at the beginning of period t is given by

$$(1-\beta)V(M_{-1}) = U(x) - x + \int_\Omega \{n\varepsilon u[q(\omega)] - (s/\alpha)c[(n/s)q(\omega)]\} f(\omega)\,d\omega$$

It is obvious that $x = x^*$ so all that remains is to choose $q(\omega)$.

The Ramsey problem facing the central bank is

$$\begin{aligned} \underset{q(\omega)}{Max} \quad & \int_\Omega \{n\varepsilon u[q(\omega)] - (s/\alpha)c[(n/s)q(\omega)]\} f(\omega)\,d\omega \\ \text{s.t.} \quad \frac{\gamma-\beta}{\beta} &= \int_\Omega \left\{ \frac{\alpha\varepsilon u'[q(\omega)]}{c'[(n/s)q(\omega)]} - 1 \right\} f(\omega)\,d\omega \end{aligned} \tag{20}$$

where the constraint facing the central bank is that the quantities chosen must be compatible with a competitive equilibrium.

If the central bank can levy lump-sum taxes, we obtain the following result.

Proposition 1 *If the central bank can levy lump-sum taxes, it chooses $i(\omega) = 0$ and $q(\omega) = q^*(\omega)$ for all states.*

According to Proposition 1, if the central bank can levy lump sum taxes then it should implement the Friedman rule $i(\omega) = 0$ for all states which can be accomplished by setting $\gamma = \beta < 1$. The reason the Friedman rule generates the first-best is that holding money is costless so agents can perfectly self-insure against consumption risks. Consequently, there are no welfare gains from stabilization policies.[10] The only friction in our model is the cost of holding money across periods. Running the Friedman rule eliminates this friction.

Now consider the case in which the central bank cannot levy lump-sum taxes, so that it is constrained to set $\gamma \geq 1$.[11] In this case we have the following result.

Proposition 2 *Without lump-sum taxes the central bank chooses $i(\omega) > 0$ and $q(\omega) < q^*(\omega)$ for all states.*

Surprisingly, in this case the central bank never chooses $i(\omega) = 0$ for any state. The reason is that the central bank wants to smooth consumption across states. Intuitively, consider two states $\omega, \omega' \in \Omega$ with $i(\omega) = 0$ implying $q(\omega) = q^*(\omega)$ and $i(\omega') > 0$ implying $q(\omega') < q^*(\omega')$. Then, the first-order loss from decreasing $q(\omega)$ is zero while there is a first-order gain from increasing $q(\omega')$. This gain can be accomplished by increasing $i(\omega)$ and lowering $i(\omega')$. Thus, the central bank's optimal policy is to smooth interest rates across states. Accordingly, unless $i(\omega) = 0$ can be done *for all* states, it is optimal to *never* set $i(\omega) = 0$. Hence, zero nominal interest rates should be an all-or-nothing policy. An interesting implication of the optimal policy is that the central bank is essentially providing an elastic supply of currency – when demand for liquidity is high, it provides additional currency and withdraws it when the demand for liquidity is low.

The ability of the central bank to smooth consumption hinges critically on the central bank's commitment to undo the state contingent injections in

[10] Ireland (1996) derives a similar result in a model with nominal price stickiness. He finds that at the Friedman rule there is no gain from stabilizing aggregate demand shocks.

[11] Another reason the central bank might be constrained from implementing the Friedman rule is that there are seigniorage needs which requires $\gamma > 1$. Since our focus is not government financing we do not model seigniorage explicitly.

market 2. To demonstrate this we now assume that the central bank does not undo the injections of the first market, i.e., $\tau_2(\omega) = 0$.

Proposition 3 *Assume that $\tau_2(\omega) = 0$ for all $\omega \in \Omega$. Then, changes in $\tau_1(\omega)$ have no real effects and any stabilization policy is ineffective.*

According to Proposition 3, if the central bank cannot commit to undo the state contingent injections of the first market stabilization policy is ineffective. The price of goods in market 1 changes proportionately with changes in $\tau_1(\omega)$. The real money holdings of the buyers are unaffected and so consumption in market 1 does not react to changes in $\tau_1(\omega)$.

In an earlier version of the paper we analyzed the case where the aggregate shocks were serially correlated. Since we have no other state variables, we were able to show that all of the above propositions continue to hold.

5 Discussion

In this section to illustrate how the optimal policy works we first present an example for which we derive closed form solutions. We then discuss why stabilization policy requires commitment to a price path by the central bank and we explore the gains from optimal stabilization.

5.1 An example

To illustrate how the optimal works consider a simple example in which the only shock is the intensive margin demand shock ε. Let ε be uniformly distributed and let preferences be given by $u(q) = 1 - \exp^{-q}$ and $c(q) = q$. With these functions the first-order conditions for the planner (26) (see Appendix) yields[12]

$$\alpha\varepsilon \exp^{-q} = \frac{n}{n - \alpha\lambda}. \tag{21}$$

Substituting this expression in the central bank's constraint we have

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \frac{\alpha\lambda}{n - \alpha\lambda} f(\varepsilon)\, d\varepsilon = \frac{\alpha\lambda}{n - \alpha\lambda}.$$

Solving for λ and substituting back into (21) yields

$$q(\varepsilon) = \ln \alpha\varepsilon - \ln(\gamma/\beta) = q^*(\varepsilon) - \ln(\gamma/\beta) \tag{22}$$

where $q(\varepsilon)$ is increasing in ε.[13] From the buyer's budget constraint

$$q(\varepsilon) = \frac{\alpha\left[1 + \tau + \tau_1(\varepsilon)\right] z}{n(1 + \tau)} \tag{23}$$

Since z is not state dependent, taking the ratios of (23) for all ε relative to $\underline{\varepsilon}$ gives

$$q(\varepsilon) = \frac{1 + \tau + \tau_1(\varepsilon)}{1 + \tau + \tau_1(\underline{\varepsilon})} q(\underline{\varepsilon}). \tag{24}$$

There is one degree of freedom in $\tau_1(\varepsilon)$ so let $\tau_1(\underline{\varepsilon}) = 0$. Thus

$$z = (n/\alpha)\left[\ln \alpha\underline{\varepsilon} - \ln(\gamma/\beta)\right]$$

and using (22) and (24) gives

$$1 + \frac{\tau_1(\varepsilon)}{1 + \tau} = \left[\frac{\ln \alpha\varepsilon - \ln(\gamma/\beta)}{\ln \alpha\underline{\varepsilon} - \ln(\gamma/\beta)}\right] > 1 \quad \text{for all } \varepsilon > \underline{\varepsilon}$$

so $\tau_1(\varepsilon) > 0$ for all $\varepsilon > \underline{\varepsilon}$.

5.2 Liquidity and inflation expectation effects

The optimal stabilization policy in our model works through a liquidity effect. For this effect to operate, the central bank must control inflation expectations

[12] With these utility and cost functions, it is easy to show that the second-order condition is satisfied.

[13] Since the Inada condition does not hold for this utility function $q(\varepsilon) = 0$ when $\gamma = \beta\alpha\varepsilon$. Thus for all $1 \leq \gamma < \beta\alpha\underline{\varepsilon}$ an equilibrium exists. For $\gamma \geq \beta\alpha\underline{\varepsilon}$ no monetary equilibrium exists.

by committing to a price path in market 2. Without commitment, injections in the first market simply change price expectations and the nominal interest rate as predicted by the Fisher equation.

To see this note from (11) that the interest rate associated with the optimal policy is

$$i(\omega) = \frac{\alpha\varepsilon u'[q(\omega)]}{c'[(n/s)q(\omega)]} - 1 > 0, \qquad \omega \in \Omega. \tag{25}$$

Assume for simplicity that the marginal cost is constant, i.e., $c'[(n/s)q(\omega)] = 1$. Then rewrite the buyers budget constraint (17) and (25) to get

$$\begin{aligned} q(\omega) &= (\alpha/n)[1+\tau+\tau_1(\omega)]\phi M_{-1} \text{ and} \\ i(\omega) &= \alpha\varepsilon u'[q(\omega)] - 1 > 0, \qquad \omega \in \Omega. \end{aligned}$$

Since the central bank has committed to a price path for ϕ, changes in $\tau_1(\omega)$ do not affect ϕ in the first equation. Hence, ϕM_{-1} is constant. It then follows that increasing $\tau_1(\omega)$ raises real balances of buyers and sellers. This decreases the real demand for loans and increases the real supply of deposits which pushes down the nominal interest rate. This in turn lowers the cost of borrowing and so raises $q(\omega)$. Consequently, state contingent injections are not neutral as long as changes in $\tau_1(\omega)$ do not affect ϕ.

What happens if the central bank cannot commit to a price path for ϕ? In the proof of Proposition 3 we show that changing $\tau_1(\omega)$ with $\tau_2(\omega) = 0$ simply increases the expected price in market 2 proportional to the increase in $\tau_1(\omega)$. Consequently, agents real money holdings remain constant. This policy increases the expected nominal interest rate. To see this note that the gross growth rate of the money supply is $\gamma_t = \tau_1(\omega) + \tau + 1$. Then substitute this and (25) into the constraint of the central bank problem to get

$$\frac{\tau_1(\omega) + \tau + 1 - \beta}{\beta} = \int_\Omega i(\omega)\, dF(\omega).$$

An increase in $\tau_1(\omega)$ increases the expected nominal interest rate. This is simply the inflation expectation effect from the Fisher equation.

5.3 The inefficiency of a passive policy

What are the inefficiencies arising from a passive policy? In order to study this question we now derive the allocation when the central bank follows a policy where the injections are not state dependent, i.e., $\tau_1(\omega) = \tau_2(\omega) = 0$, and compare it to the central bank's optimal allocation. We do so under the assumption that the central bank cannot use lump sum taxes meaning $\gamma \geq 1$. We also analyze each shock separately to understand their individual effects on the equilibrium allocation.

Extensive margin demand shocks For the analysis of shocks to n, we assume that α, ε and s are constant. Note that the optimal quantities solve $\alpha\varepsilon u'[q^*(n)] = c'[(n/s)q^*(n)]$ for all $n \in [\underline{n}, \overline{n}]$ where $q^*(n)$ is strictly decreasing in n.

Proposition 4 *For $\gamma \geq 1$, a unique monetary equilibrium exists with $q = q^*(n)$ if $n \leq \tilde{n}$ and $q < q^*(n)$ if $n > \tilde{n}$, where $\tilde{n} \in (0, \overline{n}]$. Moreover, $d\tilde{n}/d\gamma < 0$.*

With a passive policy buyers are constrained when there are many borrowers (high n) and are unconstrained when there are many depositors (low n). Since $d\tilde{n}/d\gamma < 0$, the higher is the inflation rate, the larger is the range of shocks where the quantity traded is inefficiently low. Note that for large γ we can have $\tilde{n} \leq \underline{n}$ which implies that $q < q^*(n)$ in all states.

How does this allocation differ from the one obtained by following an active policy? We illustrate the differences in Figure 2 for a linear cost function. The thick curve represents equilibrium consumption with a passive policy and the thin curve consumption when the central bank is active.

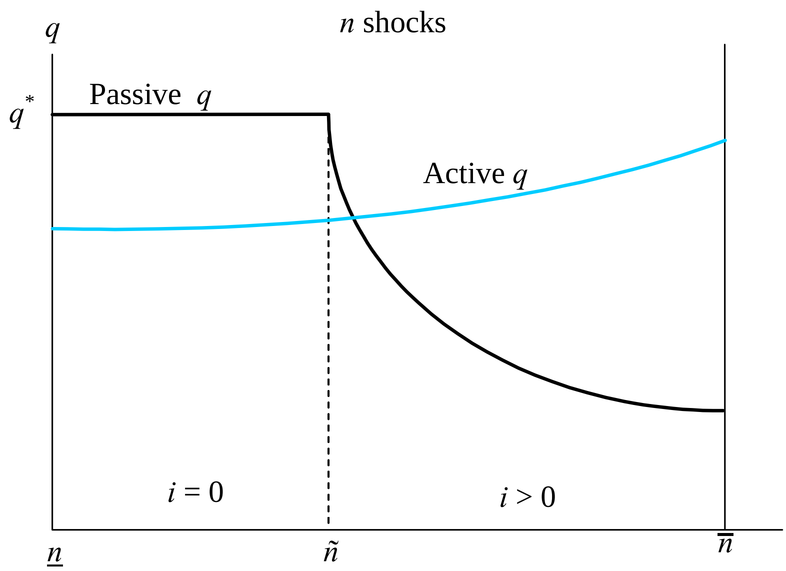

Figure 2: Shocks to the number of buyers.

As shown earlier, with an active policy buyers never consume q^* and equilibrium consumption q is increasing in n. This is just the opposite from what happens when the central bank is passive. With a passive policy, buyers consume q^* in low n states and $q < q^*$ in high n states. Moreover, q is strictly decreasing in n for $n > \tilde{n}$. These differences are also reflected in the nominal interest rates. With an active policy the nominal interest rate is strictly positive in all states and decreasing in n. In contrast, with a passive policy the nominal interest rate is $i = 0$ for $n \leq \tilde{n}$ and $i = \varepsilon\alpha u'(q) - 1 \geq 0$ for $n > \tilde{n}$, and increasing in n.

What is the role of the banking sector? With a linear cost function, in the no-banking equilibrium, the quantities consumed are the same across all states since buyers can only spend the cash they bring into market 1, which is independent of the state that is realized. In contrast, when banks exist, idle cash from sellers is deposited and lent back out to buyers. Note that individual consumption is high in low demand states and low in high demand states. The reason is that when n is high demand for loans is high and the supply of deposits is low. This pushes up the nominal interest rate and decreases individual consumption. The interesting aspect of this result is that while financial intermediation raises average consumption across states, it also causes individual consumption to fluctuate.

Intensive margin demand shocks To study ε shocks we assume that α, n and s are constant. It then follows that $\omega = \varepsilon$. Note that the optimal quantities solve $\alpha\varepsilon u'[q^*(\varepsilon)] = c'[(n/s)q^*(\varepsilon)]$where $q^*(\varepsilon)$ is strictly increasing in ε.

Proposition 5 *For $\gamma \geq 1$, a unique monetary equilibrium exists with $q < q^*(\varepsilon)$ for $\varepsilon > \tilde{\varepsilon}$ and $q = q^*(\varepsilon)$ for $\varepsilon < \tilde{\varepsilon}$, where $\tilde{\varepsilon} \in [0, \bar{\varepsilon}]$. Moreover, $d\tilde{\varepsilon}/d\gamma < 0$.*

With a passive policy, buyers are constrained in high marginal utility states but not in low states. If γ is sufficiently high, buyers are constrained in all states. Note that with a passive policy $dq/d\varepsilon > 0$ for $\varepsilon \leq \tilde{\varepsilon}$ and $dq/d\varepsilon = 0$ for $\varepsilon > \tilde{\varepsilon}$. For $\varepsilon \leq \tilde{\varepsilon}$, buyers have more than enough real balances to buy the efficient quantity. So when ε increases, they simply spend more of their money balances. For $\varepsilon > \tilde{\varepsilon}$, buyers are constrained. So when ε increases, the demand for loans increases but the supply of deposits is unchanged so no additional loans can be made. Thus, the interest rate simply increases to clear the loan market.

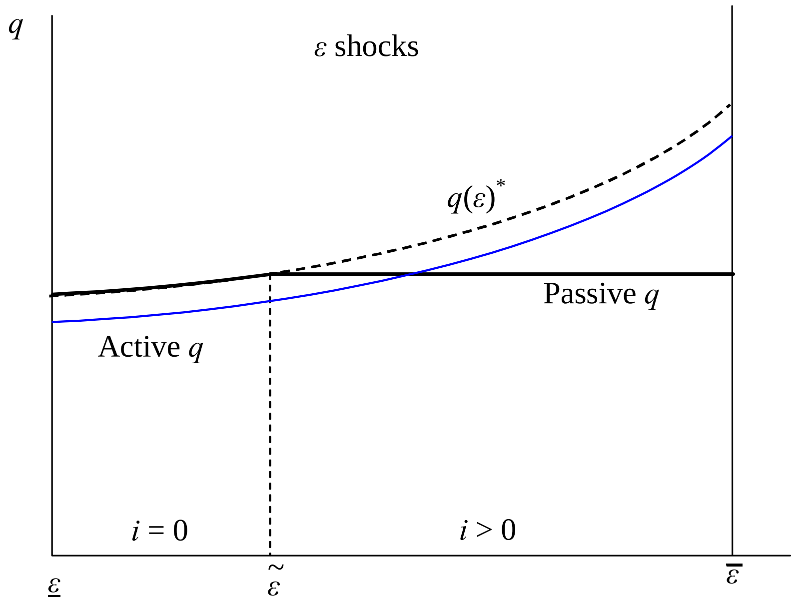

Figure 3: Marginal utility shocks.

Figure 3 illustrates how the allocation resulting from a passive policy differs from the one obtained under an active policy. The dashed curve represents the first-best quantities $q^*(\varepsilon)$. The central bank's optimal choice is strictly increasing in ε.

Finally, we have also derived the equilibrium under a passive policy for the extensive, s, and the intensive, α, supply shocks. The results and figures are qualitatively the same and we therefore do not present them here. They typically involve a cutoff value above or below the nominal interest rate is zero. These derivations are available by request.

6 Conclusion

In this paper we have constructed a dynamic stochastic general equilibrium model where money is essential for trade and prices are fully flexible. Our main result is that if the central bank commits to a long-run price path, it can successfully stabilize short-run aggregate shocks to the economy and improve welfare. The optimal policy works through a liquidity effect and involves smoothing nominal interest rates, which also smooth consumption across states. If it cannot commit to such a path, any stabilization attempts are ineffective. Monetary injections simply raise price expectations and the nominal interest rate as predicted by the Fisher equation.

There are many extensions of this model that would be interesting to pursue. For example, how would the optimal policy be affected if repayment of loans were endogenous? We have assumed that the shocks are known to the central bank. An interesting question is what is the optimal policy if the central bank has imperfect information about the nature of the aggregate shocks? A further extension would be to incorporate capital into the model to generate an intertemporal trade-off for the optimal policy. Finally, how would the existence of inside money affect the equilibrium and optimal policy. For example, would inside money eliminate the ability of the central bank to stabilize the economy? We leave this to future research.

Appendix

Lemma 1 *The marginal value of money satisfies (12).*

Proof of Lemma 1. Differentiate (6) with respect to m_1 to get

$$\begin{aligned}V'(m_1) = & \int_\Omega \Big\{ n\left[\varepsilon u'(q_b)\tfrac{\partial q_b}{\partial m_1} + W_m\left(1 - p\tfrac{\partial q_b}{\partial m_1} + \tfrac{\partial l}{\partial m_1}\right) + W_l \tfrac{\partial l}{\partial m_1}\right] \\ & +s\left[-\tfrac{c'(q_s)}{\alpha}\tfrac{\partial q_s}{\partial m_1} + W_m\left(1 + p\tfrac{\partial q_s}{\partial m_1} - \tfrac{\partial d}{\partial m_1}\right) + W_d\tfrac{\partial d}{\partial m_1}\right] \\ & (1-n-s)\left[W_m\left(1-\tfrac{\partial d}{\partial m_1}\right) + W_d\tfrac{\partial d}{\partial m_1}\right]\Big\} f(\omega)\,d\omega.\end{aligned}$$

Recall from (4) and (5) that $W_m = \phi$ and $W_d = -W_l = \phi(1+i)$ $\forall m_2$. Furthermore, $\frac{\partial q_s}{\partial m_1} = 0$ because the quantities sellers produce are independent of their money holdings and $\frac{\partial d}{\partial m_1} = 1$ since sellers and inactive agents deposit all their money holdings when $i \geq 0$. Hence,

$$\begin{aligned}V'(m_1) = & \int_\Omega \Big\{ n\left[\varepsilon u'(q_b)\tfrac{\partial q_b}{\partial m_1} + \phi\left(1 - p\tfrac{\partial q_b}{\partial m_1} + \tfrac{\partial l}{\partial m_1}\right) - \phi(1+i)\tfrac{\partial l}{\partial m_1}\right] \\ & + (1-n)\,\phi(1+i)\Big\} f(\omega)\,d\omega\end{aligned}$$

If $\omega \in \Omega_1$ we have $pq_b = m_1 + (\tau + \tau_1)M_{-1} + l$ and so $\left(1 - p\frac{\partial q_b}{\partial m_1} + \frac{\partial l}{\partial m_1}\right) = 0$. Furthermore,

$$\begin{aligned}& \varepsilon u'(q_b)\tfrac{\partial q_b}{\partial m_1} - \phi(1+i)\tfrac{\partial l}{\partial m_1} = \varepsilon u'(q_b)\tfrac{\partial q_b}{\partial m_1} - \phi(1+i)\left[p\tfrac{\partial q_b}{\partial m_1} - 1\right] \\ &= \tfrac{\partial q_b}{\partial m_1}\left[\varepsilon u'(q_b) - \phi(1+i)p\right] + \phi(1+i) = \phi(1+i) = \phi\alpha\varepsilon u'(q_b)/c'(q_s).\end{aligned}$$

If $\omega \in \Omega_0$, we have $\frac{\partial q_b}{\partial m_1} = \frac{\partial l}{\partial m_1} = 0$ since buyers are unconstrained. Hence, we get (12). ■

Proof of Proposition 1. From (20) the unconstrained optimum corresponds to $q = q^*(\omega)$ for all $\omega = \Omega$. From the constraint of the central bank problem, since $\gamma < \beta$ is feasible, the only value that is consistent with the unconstrained optimum is $\gamma = \beta$. ■

Proof of Proposition 2. The problem facing the central bank is

$$\begin{aligned}& \underset{q(\omega)}{Max} \int_\Omega \{n\varepsilon u[q(\omega)] - (s/\alpha)\,c[(n/s)\,q(\omega)]\} f(\omega)\,d\omega \\ & \text{s.t. } \frac{\gamma-\beta}{\beta} = \int_\Omega \left\{\frac{\alpha\varepsilon u'[q(\omega)]}{c'[(n/s)\,q(\omega)]} - 1\right\} f(\omega)\,d\omega\end{aligned}$$

Note that λ is independent of ω. The first-order conditions are

$$n\varepsilon u'\left[q\left(\omega\right)\right]-\left(n/\alpha\right)c'\left[\left(n/s\right)q\left(\omega\right)\right]+\lambda\Psi\left(\omega\right)=0\qquad\omega\in\Omega,\tag{26}$$

where

$$\Psi\left(\omega\right)=\alpha\varepsilon\left\{\frac{u''\left[q\left(\omega\right)\right]c'\left[\left(n/s\right)q\left(\omega\right)\right]-\left(n/s\right)c''\left[\left(n/s\right)q\left(\omega\right)\right]u'\left[q\left(\omega\right)\right]}{c'\left[\left(n/s\right)q\left(\omega\right)\right]^{2}}\right\}<0$$

Sufficient conditions for a maximum are $u'''\left[q\left(\omega\right)\right]\geq 0\geq c'''\left[\left(n/s\right)q\left(\omega\right)\right]$ for all $\omega\in\Omega$. The rest of the proof immediately follows from inspecting the first-order conditions (26). ■

Proof of Proposition 3. In any equilibrium buyers' real money holdings are

$$\phi\left(\omega\right)\left\{M_{-1}\left[1+\tau+\tau_{1}\left(\omega\right)\right]+l\left(\omega\right)\right\}=\frac{\phi\left(\omega\right)M_{-1}\left[1+\tau+\tau_{1}\left(\omega\right)\right]}{n}=\frac{\phi\left(\omega\right)M\left(\omega\right)}{n}$$

since the end-of-period nominal money stock is $M\left(\omega\right)=M_{-1}\left[1+\tau+\tau_{1}\left(\omega\right)\right]$. Thus, in any equilibrium we must have

$$\phi\left(\omega\right)p\left(\omega\right)q\left(\omega\right)\leq\frac{\phi\left(\omega\right)M\left(\omega\right)}{n}\qquad\text{for all }\omega\in\Omega.$$

The first-order conditions of the sellers (7) imply

$$\alpha^{-1}c'\left[\left(n/s\right)q\left(\omega\right)\right]q\left(\omega\right)\leq\frac{\phi\left(\omega\right)M\left(\omega\right)}{n}\qquad\text{for all }\omega\in\Omega.$$

In a steady-state equilibrium $\phi\left(\omega\right)M\left(\omega\right)=\phi_{-1}\left(\omega\right)M_{-1}\left(\omega\right)=z\left(\omega\right)$ for all $\omega\in\Omega$. Hence,

$$\alpha^{-1}c'\left[\left(n/s\right)q\left(\omega\right)\right]q\left(\omega\right)\leq\frac{z\left(\omega\right)}{n}\qquad\text{for all }\omega\in\Omega.\tag{27}$$

We now show that in any steady state equilibrium $z\left(\omega\right)=z$ is a constant. To do so rewrite (12) as follows

$$V'(m_{1})=\int_{\Omega}\left[\phi\left(\omega\right)\alpha\varepsilon u'\left[q_{b}\left(\omega\right)\right]/c'\left[q_{s}\left(\omega\right)\right]\right]f\left(\omega\right)d\omega.$$

Use (3) to eliminate $V'(m_1)$ and (14) to eliminate q_s to get

$$\phi_{-1}(\omega_{-1})/\beta = \int_{\Omega} [\phi(\omega)\,\alpha\varepsilon u'[q(\omega)]/c'[(n/s)\,q(\omega)]]\,f(\omega)\,d\omega.$$

Multiply this expression by $M_{-1}(\omega_{-1})$ to get

$$M_{-1}(\omega_{-1})\,\phi_{-1}(\omega_{-1})/\beta = \int_{\Omega} \left\{ \frac{M(\omega)\,\phi(\omega)}{\gamma(\omega)} \frac{\alpha\varepsilon u'[q(\omega)]}{c'[(n/s)\,q(\omega)]} \right\} f(\omega)\,d\omega.$$

since $M(\omega) = [1 + \tau + \tau_1(\omega)]\,M_{-1}(\omega_{-1}) = \gamma(\omega)\,M_{-1}(\omega_{-1})$. Note that in any steady-state equilibrium the right-hand side is independent of ω_{-1} and therefore a constant. This immediately implies that

$$M_{-1}(\omega_{-1})\,\phi_{-1}(\omega_{-1}) = z_{-1} = constant \qquad \text{for all } \omega_{-1} \in \Omega$$

Since in a steady state equilibrium we have $z_{-1} = z$ we can rewrite this equation as follows

$$1/\beta = \int_{\Omega} \left\{ \frac{\alpha\varepsilon u'[q(\omega)]}{\gamma(\omega)\,c'[(n/s)\,q(\omega)]} \right\} f(\omega)\,d\omega.$$

Finally from (27) we have

$$\alpha^{-1} c'[(n/s)\,q(\omega)]\,q(\omega) \leq \frac{z}{n} \qquad \text{for all } \omega \in \Omega.$$

Since the right-hand side is independent of $\gamma(\omega)$, changes is $\gamma(\omega)$ are neutral. Hence, stabilization policy is ineffective. ■

We use Lemma 2 in the proofs of Propositions 4 and 5.

Lemma 2 *Under efficient trading, real aggregate spending* $n\phi p(\omega)\,q^*(\omega)$ *is increasing in* ε*. It is increasing in* n *and decreasing in* s *and* α *if*

$$\Phi = 1 + \frac{q^* u''(q^*)}{u'(q^*)} - \frac{q^* c''[(n/s)q^*]\,(n/s)}{c'[(n/s)\,q^*]} < 0$$

Proof of Lemma 2. In equilibrium buyer's real money holdings are $(\upsilon/n)\,z$. Thus, in any equilibrium $n\phi pq \leq \upsilon z$. The right-hand side is the aggregate real money stock in market 1 which is independent of ω. The left-hand side is real aggregate spending which is a function of ω. For a given state ω, trades are efficient if $n\phi p(\omega) q^*(\omega) \leq \upsilon z$ and inefficient if $n\phi p(\omega) q^*(\omega) > \upsilon z$ where $p = p(\omega)$ is a function of ω but ϕ is not. We would like to know how real aggregate spending $g(\omega) = n\phi p(\omega) q^*(\omega)$ changes in ω when trades are efficient:

$$dg(\omega) = \phi p(\omega) q^*(\omega) dn + n\phi q^*(\omega) dp + n\phi p(\omega) dq^*$$

The first term reflects the change in real liquidity that is intermediated in the economy. This effect only occurs if n changes. The second term reflects changes in the relative price ϕp of goods and the third term changes in the efficient quantity. Rewrite it as follows

$$dg(\omega) = n\phi p q^* \left[\frac{dn}{n} + \frac{dp}{p} + \frac{dq^*}{q^*}\right]$$

The term $\frac{dp}{p}$ can be derived from (7) as follows

$$\frac{dp}{p} = \frac{c''\left[(n/s) q^*\right]}{c'\left[(n/s) q^*\right]} \frac{q^* n}{s} \left[\frac{dn}{n} - \frac{ds}{s}\right] - \frac{d\alpha}{\alpha}$$

and the term $\frac{dq^*}{q^*}$ can be derived from $\varepsilon\alpha u'(q^*) = c'\left[(n/s)q^*\right]$ as follows

$$\begin{aligned} \frac{dq^*}{q^*} &= \frac{c''\left[(n/s)q^*\right](n/s)}{\alpha\varepsilon u''(q^*) - c''\left[(n/s)q^*\right](n/s)} \left[\frac{dn}{n} - \frac{ds}{s}\right] \\ &\quad - \frac{\varepsilon\alpha u'(q^*)/q^*}{\alpha\varepsilon u''(q^*) - c''\left[(n/s)q^*\right](n/s)} \left[\frac{d\alpha}{\alpha} + \frac{d\varepsilon}{\varepsilon}\right] \end{aligned}$$

Investigating each shock separately we get

$$\begin{aligned} \frac{\partial g(n)}{\partial n} &= c'\left[(n/s)q^*\right] q^*(n/\alpha s) \left\{1 + \frac{c''\left[(n/s) q^*\right]\Phi}{\alpha\varepsilon u''(q^*) - c''\left[(n/s)q^*\right](n/s)}\right\} \geq 0 \\ \frac{\partial g(n)}{\partial s} &= -\frac{c'\left[(n/s)q^*\right] q^*(n/s)^2 c''\left[(n/s) q^*\right]\Phi}{\alpha\left[\alpha\varepsilon u''(q^*) - c''\left[(n/s)q^*\right](n/s)\right]} \leq 0 \\ \frac{\partial g(n)}{\partial \alpha} &= \frac{-c'\left[(n/s)q^*\right] n\varepsilon u'(q^*)\Phi}{\alpha\left[\alpha\varepsilon u''(q^*) - c''\left[(n/s)q^*\right](n/s)\right]} < 0 \\ \frac{\partial g(n)}{\partial \varepsilon} &= -\frac{c'\left[(n/s)q^*\right] n u'(q^*)}{\alpha\varepsilon u''(q^*) - c''\left[(n/s)q^*\right](n/s)} > 0 \end{aligned}$$

$\blacksquare$

Proof of Proposition 4. Here $\omega = n$. From (19) we have

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{n}}^{\overline{n}} \left[\frac{\alpha \varepsilon u' \left(q\left(n, z\right)\right)}{c' \left[\left(n/s\right) q\left(n, z\right)\right]} - 1 \right] f\left(n\right) dn. \tag{28}$$

Lemma 2 gives $\frac{\partial g(n)}{\partial n} \geq 0$. If $g\left(\underline{n}\right) > \upsilon z$, then agents are constrained in all states. If $g\left(\overline{n}\right) < \upsilon z$, then agents are never constrained. If $g\left(\overline{n}\right) \geq \upsilon z \geq g\left(\underline{n}\right)$, for a given value of z there is a unique critical value $\tilde{n}$ such that

$$g\left(\tilde{n}\right) = \upsilon z \tag{29}$$

This implies that $q = q^*\left(n\right)$ for $n \leq \tilde{n}$ and $q < q^*\left(n\right)$ for $n > \tilde{n}$. Note that $\frac{\partial \tilde{n}}{\partial z} \geq 0$.

Existence: Using (29) we can write (28) as follows

$$\frac{\gamma - \beta}{\beta} = \int_{\check{n}}^{\overline{n}} \left\{ \frac{\alpha \varepsilon u' \left[q\left(n, z\right)\right]}{c' \left[\left(n/s\right) q\left(n, z\right)\right]} - 1 \right\} f\left(n\right) dn \equiv RHS \tag{30}$$

where $\check{n} = \max\left\{\tilde{n}, \underline{n}\right\}$. Only the right-hand side is a function of z. Note that $\lim\limits_{z \to 0} RHS = \infty$. For $\upsilon \overline{z} = g\left(\overline{n}\right)$ we have $\tilde{n} = \overline{n}$ and therefore $RHS\left|_{z = \overline{z}}\right. = 0 \leq \frac{\gamma - \beta}{\beta}$. Since RHS is continuous in z an equilibrium exists.

Uniqueness: The right-hand side of (30) is monotonically decreasing in z. To see this use Leibnitz's rule to get

$$\begin{aligned} \frac{\partial RHS}{\partial z} &= \int_{\check{n}}^{\overline{n}} \frac{\alpha \varepsilon \left[u'' c' - \left(n/s\right) c'' u'\right]}{\left(c'\right)^2} \frac{\partial q\left(n, z\right)}{\partial z} f\left(n\right) dn \\ &\quad - \left\{ \frac{\alpha \varepsilon u' \left[q\left(\check{n}, z\right)\right]}{c' \left[\left(\check{n}/s\right) q\left(\check{n}, z\right)\right]} - 1 \right\} f\left(\check{n}\right) \frac{\partial \check{n}}{\partial z} \end{aligned}$$

Since $q\left(\check{n}, z\right) = q^*\left(\check{n}\right)$ by construction we have

$$\frac{\partial RHS}{\partial z} = \int_{\check{n}}^{\overline{n}} \frac{\alpha \varepsilon \left[u'' c' - \left(n/s\right) c'' u'\right]}{\left(c'\right)^2} \frac{\partial q\left(n, z\right)}{\partial z} f\left(n\right) dn < 0.$$

Since the right-hand side is decreasing in z, we have a unique z that solves (30). Consequently, we have

$$q = q^*\left(n\right) \text{ if } n \leq \check{n} \text{ and } q < q^*\left(n\right) \text{ otherwise.}$$

Finally, if buyers have been constrained in market 1 money holdings at the opening of the second market are $m_2 = 0$ for buyers and inactive agents and $m_2 = pq_s$ for sellers. Solving for equilibrium consumption and production in the second market, with $x^* = U'^{-1}(1)$, gives

$$
\begin{aligned}
h_b &= x^* + ne_c(q)\, c\left[(n/s)\, q\right] + (1-n)\, e_u(q)\, u(q) \\
h_s &= x^* - ne_c(q)\, c\left[(n/s)\, q\right](1-s)\, s^{-1} - ne_u(q)\, u(q) \\
h_{in} &= x^* + ne_c(q)\, c\left[(n/s)\, q\right] - ne_u(q)\, u(q)
\end{aligned}
$$

Notice that $nh_b + sh_s + (1-n-s)\, h_{in} = x^*$. Moreover, we have $h_b \geq h_{in} \geq h_s$. For existence we need that all agents work a positive amount in the second market. This, it is sufficient to show that $h_s > 0$.

Given $\underline{s} > 0$, n/s is bounded and since the elasticities $e_c(q)$ and $e_u(q)$ are bounded, we can scale $U(x)$ such that there is a value $x^* = U'^{-1}(1)$ greater than the last term for all $q \in [0, q^*]$. Hence, h_s is positive for for all $q \in [0, q^*]$ ensuring that the equilibrium exists. Note that the states where the buyers are constrained are the ones where the sellers have all the money after trading. Therefore, if h_s is positive in constrained states it is positive in all unconstrained states. ■

Proof of Proposition 5. Here $\omega = \varepsilon$. From (19) we have

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \left[\frac{\alpha \varepsilon u'\left[q(\varepsilon, z)\right]}{c'\left[(n/s)\, q(\varepsilon, z)\right]} - 1 \right] f(\varepsilon)\, d\varepsilon. \tag{31}$$

Lemma 2 gives $\frac{\partial g(\varepsilon)}{\partial \varepsilon} \geq 0$. If $g(\underline{\varepsilon}) > vz$, then agents are constrained in all states. If $g(\overline{\varepsilon}) < vz$, then agents are never constrained. If $g(\overline{\varepsilon}) \geq vz \geq g(\underline{\varepsilon})$, for a given value of z there is a unique critical value $\tilde{\varepsilon}$ such that

$$g(\tilde{\varepsilon}) = vz \tag{32}$$

This implies that $q = q^*(\varepsilon)$ for $\varepsilon \leq \tilde{\varepsilon}$ and $q < q^*(\varepsilon)$ for $\varepsilon > \tilde{\varepsilon}$. Note that $\frac{\partial \tilde{\varepsilon}}{\partial z} \geq 0$.

Existence: Using (32) we can write (31) as follows

$$\frac{\gamma-\beta}{\beta}=\int_{\check{\varepsilon}}^{\overline{\varepsilon}}\left\{\frac{\alpha\varepsilon u'\left[q\left(\varepsilon,z\right)\right]}{c'\left[\left(n/s\right)q\left(\varepsilon,z\right)\right]}-1\right\}f\left(\varepsilon\right)d\varepsilon\equiv RHS \tag{33}$$

where $\check{\varepsilon}=\max\left\{\tilde{\varepsilon},\underline{\varepsilon}\right\}$. Only the right-hand side is a function of z. Note that $\lim_{z\to 0}RHS=\infty$. For $\upsilon\overline{z}=g\left(\overline{\varepsilon}\right)$ we have $\tilde{\varepsilon}=\overline{\varepsilon}$ and therefore $RHS\left|_{z=\overline{z}}\right.=0\leq\frac{\gamma-\beta}{\beta}$. Since RHS is continuous in z an equilibrium exists.

Uniqueness: The right-hand side of (33) is monotonically decreasing in z. To see this use Leibnitz's rule and not that by construction $q\left(\check{\varepsilon},z\right)=q^{*}\left(\check{\varepsilon}\right)$ to get

$$\frac{\partial RHS}{\partial z}=\int_{\check{\varepsilon}}^{\overline{\varepsilon}}\frac{\alpha\varepsilon\left[u''c'-\left(n/s\right)c''u'\right]}{\left(c'\right)^{2}}\frac{\partial q\left(\varepsilon,z\right)}{\partial z}f\left(\varepsilon\right)d\varepsilon<0.$$

Since the right-hand side is strictly decreasing in z, we have a unique z that solves (33). Consequently, we have

$$q=q^{*}\left(\varepsilon\right)\ \text{ if }\varepsilon\leq\check{\varepsilon}\text{ and }q<q^{*}\left(\varepsilon\right)\ \text{ otherwise.}$$

Finally, it is straightforward to show that the hours worked in market 2 are bounded away from zero. ■

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