Credit Risk in General Equilibrium

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Credit risk models used in quantitative risk management treat credit risk analysis conceptually like a single person decision problem. From this perspective an exogenous source of risk drives the fundamental parameters of credit risk: probability of default, exposure at default and the recovery rate. In reality these parameters are the result of the interaction of many market participants: They are endogenous. We develop a general equilibrium model with endogenous credit risk that can be viewed as an extension of the capital asset pricing model. We analyze equilibrium prices of securities as well as equilibrium allocations in the presence of credit risk. We use the model to discuss the conceptual underpinnings of the approach to risk weight calibration for credit risk taken by the Basel Committee.

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1. Introduction

During the past two decades credit risk modeling flourished both in the academic literature and in the financial industry (see McNeil et al. [2005] for an overview). Credit risk modeling also had a considerable influence on recent developments in bank capital regulation. The calibration of risk weights for different asset classes on a bank's balance sheet under the 2001 reform of the Basel Accord of 1988 (Basel II) was guided by recent developments in credit risk modeling.

What is common to most of the credit risk models used in risk management and regulation is the assumption that individual characteristics of credit instruments, in particular the probability of default, the exposure at default and the recovery rate, follow some exogenous probability law that can be estimated using historical data. The credit risk model maps these characteristics into a loss distribution over a fixed time horizon. Loss distributions of loan portfolios derived in this way are often used in risk management to quantify the size of equity buffers necessary to support the portfolio. In capital adequacy regulation under Basel II this concept of a loss distribution for a portfolio of credit instruments is used to calibrate risk weights imposed by the regulator.

From an economic perspective, the probability of default, the exposure at default and the loss given default are more naturally thought of as the aggregate result of individual behavior. Thus, it makes more sense to think of credit risk as endogenous.

Thinking of credit risk as endogenous opens a different perspective that is in sharp contrast to the prevailing analysis of credit risk. While standard credit risk modeling thinks of credit risk analysis as a decision problem under risk, the endogenous view of credit risk is conceptually nearer to an equilibrium phenomenon resulting from interacting individual decisions of many agents. From such a perspective the standard applications of credit risk modeling in regulation immediately look problematic. In the application of credit risk models, the economic capital that is required to support a given portfolio is derived as some quantile of the loss distribution, usually the value-at-risk. If credit risk is endogenous there is a feedback effect between economic capital and the loss distribution. Capital requirements will change the behavior of individuals and thus the loss distribution and the risk weights which will in turn influence behavior. But not only in the cross section also over time endogenous credit risk leads to other problems like pro-cyclical effects of capital requirements, a problem that has been widely discussed in the literature and the policy debate and that has been addressed in the recent amendments to the Basel II framework.[1] If credit risk is indeed endogenous, this has wider implications for regulatory reform that go beyond the refinement of calibration of risk weights for different asset classes.

In this paper we make an attempt to analyze credit risk as an equilibrium phenomenon rather than as a decision problem under risk. Equilibrium analysis takes the endogenous credit risk view to its extreme. We hope that for a conceptual discussion this extreme turn in perspective is useful. Our aim is thus to formulate and analyze an equilibrium

[1] The debate on procyclicality has accompanied the Basel process from the beginning. Important references are Hellwig and Blum [1995], Danielsson et al. [2001] and Shin [2010] and the references given there.

model of credit risk from which we can learn something about the limits of standard credit risk modeling in banking regulation.

An Overview of the Model We analyze credit risk by an abstract model of competitive borrowing and lending with the possibility of bankruptcy. The exogenous parameters in our analysis are state probabilities, risk preferences of individuals, endowments, financial instruments and bankruptcy rules. What is endogenously determined by the model are prices and allocations of risky securities including credit instruments as well as the allocation of consumption indirectly induced by this allocation. These consumption plans determine the exposure at default, default states and recovery rates and thus the parameters of credit risk, assumed to be exogenous in traditional credit risk analysis. What the model thus provides is an abstract perspective on a world where credit risk is endogenous which can then be contrasted with a world where credit risk is exogenous. The abstract perspective is able to highlight the basic logic of endogenous credit risk but hides much of the institutional structure, such as the operation of borrowing and lending through a banking system. While this feature makes the model not very useful to discuss the specific design of capital regulation for banks we believe that it provides clear perspective on the possibilities and limits of risk weight calibration based on traditional credit risk models and thus on the conceptual underpinnings of current capital regulation for banks.

Related Research Our model builds on the literature on default in general equilibrium pioneered by Zame [1993] and Dubey et al. [2005] and developed in various variations in Modica et al. [1998], Araujo and Pascoa [2002] and Sabarwal [2003]. This literature in principle provides a framework that allows for an abstract analysis of endogenous credit risk. For the current analysis it is however difficult to directly draw on these papers. For our purpose they are too abstract because they are almost entirely focused on the equilibrium concept and on existence results. They are also too general to clearly focus on the specific aspect of endogenous credit risk.

Building on this literature we aim at a general equilibrium model of bankruptcy that is more general than a fully parametrized example but specific enough to allow for a structured discussion of equilibrium prices and allocations. The idea is thus to go beyond a pure existence result by providing enough structure to say more about equilibrium and its main properties. Our leading example for this kind of analysis are Magill and Quinzii [1997] and Magill and Quinzii [2000] as well as the exposition of the CAPM in Geanakoplos and Shubik [1990] and in Magill and Quinzii [1995]. In the context of general equilibrium modeling of default, our contribution is a model that is structurally similar to the consumption based CAPM and contains the CAPM as a special case for the limiting case of no default. It therefore allows to look at the general equilibrium modeling of default from the perspective of a very well understood framework of financial economics. By this formulation it also opens an opportunity to expose general equilibrium models of default to experimental tests along the lines of Bossaerts [2002] and Bossaerts et al. [2007]. We hope that the specific CAPM-like formulation of a general equilibrium model of bankruptcy and default is next to our main focus on the endogeneity of credit risk an interesting additional contribution to the literature on default in general equilibrium.

For the case of market risk the nature and consequences of endogenous risk, risk that arises by pricing and repricing in financial markets, have been analyzed in Danielsson et al. [2009] and Shin [2010]. The institutional structure of intermediation and bank balance sheets are directly built into these models. Our analysis provides a complementary perspective on endogenous risk for the case of credit risk.

Structure of the paper We begin in section (2) with the analysis of a simple example of competitive borrowing and lending that illustrates the main concepts and idea of our analysis of credit risk in general equilibrium. In section (3) we describe and analyze the model, we define the concept of bankruptcy equilibrium. In section (4) we present the central results. We prove existence and characterize the properties of equilibrium prices and quantities. Section (5) clarifies the relations between the CAPM and our model of bankruptcy equilibrium. Section (6) analyses the implications of endogenous credit risk for credit risk modeling and risk calibration. Finally section(7) concludes. An appendix contains proofs of propositions.

2. Bankruptcy Equilibrium: An Example

Let us start with a simple example of competitive borrowing and lending. In the context of this example we can develop the basic elements and arguments behind our approach to modeling bankruptcy in general equilibrium and introduce some of our basic concepts as well as some notation.

In our example two risk averse agents live for one period starting today ($t = 0$) and ending tomorrow ($t = 1$). They have endowments of a consumption good today and tomorrow. The endowments are described by the vectors $\boldsymbol{\omega}^1 = (\omega_0^1, \omega_1^1)$ and $\boldsymbol{\omega}^2 = (\omega_0^2, \omega_1^2)$ with all entries positive. Both agents have standard preferences for consumption $\boldsymbol{x}^i = (x_0^i, x_1^i)$ that can be described by a utility function $u^i(\boldsymbol{x}^i)$.

The agents can achieve an inter-temporal consumption profile by buying or selling today a bond, which promises one unit of consumption good tomorrow in quantities z^i at price q on a competitive financial market. Borrowing and lending in this model is competitive in the sense that agents maximize their utility function within their budget constraint taking the price of the bond q as given.

To contrast the model with default with a model without default, it might be helpful to look at a picture which shows a competitive equilibrium with borrowing and lending in a net-trade diagram.

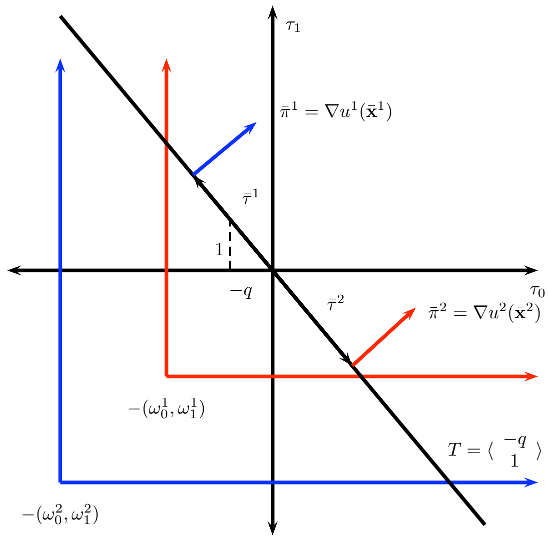

Figure 1: Competitive borrowing and lending in equilibrium without default. Optimality of decisions requires that the utility gradients π^i of the lender and the borrower are orthogonal to the space of net income transfers T spanned by the possibility to trade a bond. The equilibrium problem is then to find a price q that rotates the space of net income transfers in a way that the net transfers τ^i add up to zero.

Let us now introduce bankruptcy into the example. By bankruptcy we mean a situation where an agent is not anymore able to honor his debts $\boldsymbol{z}^i$ given the value of his assets ω_1^i at $t = 1$. Note that we do not allow for default: As long as the debt $\boldsymbol{z}^i$ can be paid given the asset value ω_1^i the agent is obliged to honor his obligation. Bankruptcy means allowing agents to *plan* a negative consumption tomorrow. For given preferences this might be the optimal choice.

If we give agents this option, why will they find an optimal amount of borrowing at all? Can't they just increase their $t = 0$ consumption very much by borrowing and then going bankrupt? The answer is that this strategy is limited by a cost to bankruptcy. We assume that the utility function evaluates *jointly* positive *and* negative consumption plans.

Assume that our utility function embeds a *penalty function* which is strictly increasing in the value of the planned shortfall in a bankruptcy:[2]

$$x^{i-} := 0 \wedge (\omega_1^i + \boldsymbol{z}^i)$$

If we denote the consumption value corresponding to a non-negative value of assets in terms of the consumption good, by

$$x^{i+} = 0 \vee (\omega_1^i + \boldsymbol{z}^i)$$

[2]We use the notation $\vee$ and $\wedge$ as the maximum and minimum operator. Applied to vectors the operators give the component-wise maximum of minimum.

we have

$$\begin{aligned} x_1^i &= x_1^{i+} + x_1^{i-} \\ &= 0 \wedge (\omega_1^i + \boldsymbol{z}^i) + 0 \vee (\omega_1^i + \boldsymbol{z}^i) \\ &= \omega_1^i + \boldsymbol{z}^i. \end{aligned}$$

The idea to model the costs of default as a utility penalty is due to Dubey et al. [2005] and Zame [1993]. In our setup the consumer ex ante evaluates a consumption plan with respect to *both* the real consumption x_s^{i+} and the default penalty x_s^{i-} such that $u^i(x_0^i, x_1^i) = u^i(x_0^1, (x_1^{i+} + x_1^{i-}))$. In Dubey et al. [2005] and Zame [1993] $u^i(x_0^i, x_1^i)$ is separated into the sum $u^i(x_0^i, x_1^{i+}) + \lambda w^i(x_1^{i-})$, the utility from real consumption and the penalty function. In our setup the utility function evaluates real consumption and default penalty jointly.[3]

But how can planned bankruptcy occur in equilibrium? To see how, assume agent 2 is the lender and agent 1 is the borrower. Let us look at the problem of agent 2 first. Every unit of lending provided to the market has to be discounted because only a fraction $r \in [0, 1]$ will be recovered. His problem is now to find an optimum consumption plan by taking this recovery rate into account. The recovery rate that will emerge depends on the decision of agent 1: The *recovery rate* is given by

$$r = \frac{-\boldsymbol{z}^1 \wedge \omega_1^1}{-\boldsymbol{z}^1}$$

It is the fraction of what is actually paid given to what payment has been promised.

Agent 1, the borrower, takes the payoff profile of the the bond (in which he holds a short position) as given. He does not have to take into account potential bankruptcy by others, since he is borrowing.

The equilibrium problem now changes in the following way compared with a situation without bankruptcy: The optimization problem looks different from the perspective of borrowers and lenders. From the perspective of agent 1, the borrower, the problem is to find a net income transfers $\boldsymbol{\tau}^1$ in the cone spanned by the vector $T_s = (q, -1)$ such that his utility gradient $\boldsymbol{\pi}^1 = \nabla u^1(\bar{\boldsymbol{x}}^1)$ is orthogonal to the ray spanned by T_s. For agent 2 the situation is different: He has to find all net income transfers in the cone spanned by the vector $T_l = (-q, r)$ such that his utility gradient $\boldsymbol{\pi}^2 = \nabla u^2(\bar{\boldsymbol{x}}^2)$ is orthogonal to the ray spanned by T_l. The decision of agent 1 takes the default penalty into account, since the real consumption he can get at time $t = 1$ is only zero. The distance between his consumption plan and his equilibrium consumption is exactly the bankruptcy penalty.

The equilibrium problem is now more involved than in a world without bankruptcy since *both* q and r have to adjust to guarantee that financial markets clear and that the non-negative parts of the consumption plans are equal to the available resources. The equilibrium problem is now to find a $(\bar{q}, \bar{r})$ such that both cones T_l and T_s make consumption compatible with the resource constraints in the economy.

[3] At a technical level this approach in combination with considering bankruptcy (in contrast to default) makes the individual decision problem tractable because linearity of the constraints is preserved at the level of the individual decision problem.

Formally $(\bar{q}, \bar{\boldsymbol{r}})$ is a *bankruptcy equilibrium* when agents have taken an optimal decision and

$$(\boldsymbol{\tau}^1(\bar{q}, \bar{\boldsymbol{r}})) \vee -\boldsymbol{\omega}^1) + (\boldsymbol{\tau}^2(\bar{q}, \bar{\boldsymbol{r}}) \vee -\boldsymbol{\omega}^2) = 0$$

To interpret the equilibrium condition in economic terms we rewrite it: First, since agents must not choose a negative consumption in $t = 0$, security markets must clear or[4]

$$\boldsymbol{z}^1(\bar{q}, \bar{\boldsymbol{r}}) + \boldsymbol{z}^2(\bar{q}, \bar{\boldsymbol{r}}) = 0$$

Furthermore $(\bar{q}, \bar{\boldsymbol{r}})$ must be such that

$$(x^1(\bar{q}, \bar{\boldsymbol{r}}) \vee 0) + (x^2(\bar{q}, \bar{\boldsymbol{r}}) \vee 0) = \omega_1^1 + \omega_1^2$$

The condition that date 1 consumption must be feasible is equivalent to requiring that the expected recovery rate is equal to the recovery rate that is realized at $t = 1$. The equilibrium condition could therefore be alternatively formulated as requiring security markets to clear and that agents correctly expect the equilibrium recovery rate $\bar{r}$.

In this simple example a bankruptcy equilibrium can be visualized. Figure 2 displays the basic geometry of a bankruptcy equilibrium. Contrary to the no bankruptcy case, the space of net income transfers gets a "kink". Lenders and borrowers live financially in two different worlds and a wedge between promises sold and promises purchased arises. Still if agents hold rational expectations about the equilibrium recovery rate, prices and the recovery rate can adjust to let security markets clear at one price q and keep the aggregate resource constraints in the economy. From the picture we see that it can occur that the bond market might not be used by all agents. More generally, with more agents there might be agents who are not trading in the bond in equilibrium. The higher the credit risk of the bond (the lower is r) the lower is market and funding liquidity in the bond. In our model credit and liquidity risk are two sides of the same coin.

In the picture we see that agent 1 finds it optimal to chose a consumption plan which implies a bankruptcy tomorrow. His transfer cone T_s is given by the dashed red ray starting at the origin of the net transfer space. At this consumption plan he has to pay a penalty equal to the vertical distance between his optimal choice and the zero consumption line.

From the picture we see how the bond market can equilibrate in such a situation. The transfer space of the lender has to be rotated from the ray $(-q, 1)$ to $T_l = (-q, r)$ such that if this ray is prolonged to an imaginary linear space it passes through the zero consumption line of agent 1 at $x_1^1 = 0$. In this position the market can clear because at this point the *actual* net income transfers sum to zero and thus $(\boldsymbol{\tau}^1(\bar{q}, \bar{\boldsymbol{r}})) \vee -\boldsymbol{\omega}^1) + (\boldsymbol{\tau}^2(\bar{q}, \bar{\boldsymbol{r}}) \vee -\boldsymbol{\omega}^2) = 0$.

This example reveals in a nutshell the basic mechanics of a bankruptcy equilibrium. But where is credit risk entering the picture? This aspect of bankruptcy can only be seen in a slightly more complex version of the example, which we are going to develop in our model. In the more complex version there are different states of the world at $t = 1$ each occurring with some given probability. As in the simple example agents

[4]This condition is equivalent to $\tau_0^1 + \tau_0^2 = 0$.

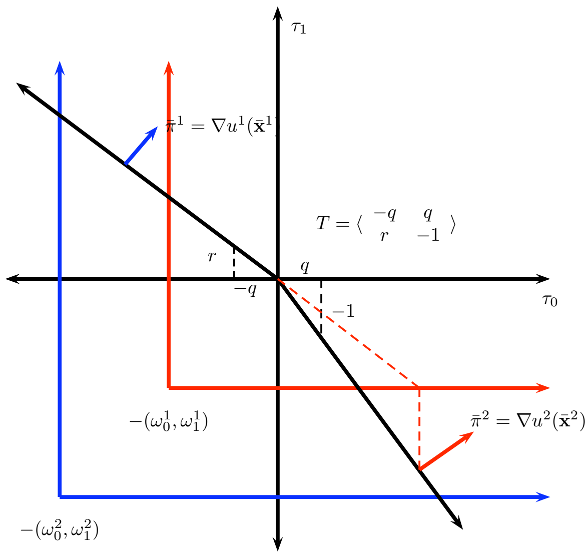

Figure 2: Competitive borrowing and lending with bankruptcy. What was the net transfer space before now gets a kink. Optimality for the lender requires that his utility gradient π^2 is orthogonal to the ray spanned by the vector $T^l = (-q, r)$. For the borrower the utility gradient π^1 is orthogonal to the ray spanned by $T^s = (q, -1)$ because he holds a short position in the bond. The default penalty is equal to the vertical distance of his optimal decision point to the point where his date one consumption is zero. The equilibrium problem is that q and r have to adjust such that the linear space spanned by $T^l = (-q, r)$ passes through the time 1 zero consumption point of agent 1. At this point the real net transfers are such that they add up to zero.

can use a financial instrument with which they can borrow and lend, possibly with a planned bankruptcy in some state of the world. Agents can also use alternative financial instruments which may be used for risk sharing only. In this slightly more complicated version agents take an optimal portfolio decision and financial instruments are priced according to their risk characteristics. Clearly the decision of some agents to choose a consumption plan that implies bankruptcy in some state is a credit risk from the viewpoint of the lenders. But this is a risk that arises endogenously as a consequence of the agents' decisions. All parameters of credit risk that are assumed to follow an exogenous probability law in the usual credit risk models, the probability of default, the exposure at default and the recovery rate are now endogenously determined in equilibrium.

3. The Model

3.1. A Bond Equity Economy

We consider a pure exchange economy with one commodity and a finite number I of agents. There are two dates, $t = 0$ and $t = 1$, and a finite number S of states of the world at date $t = 1$ which occur with probabilities $\rho_s \neq 0$.

Each agent is characterized by a consumption set $X \subset \mathbb{R}^{S+1}$, a utility function $u^i : X \to \mathbb{R}$ and an initial endowment $\boldsymbol{\omega}^i = (\omega_0^i, \omega_{\mathbf{1}}^i) \in X \cap \mathbb{R}_{++}^{S+1}$. We denote ω_0^i is the endowment at $t = 0$ and $\omega_{\mathbf{1}}^i = (\omega_1^i, \ldots, \omega_S^i)$ is the endowment vector at $t = 1$. In a similar manner we denote by $\boldsymbol{x}^i = (x_0^i, x_{\mathbf{1}}^i) \in X$ the consumption plan of agent i. We denote $\omega_{\mathbf{1}} = \sum_{i=1}^{I} \omega_{\mathbf{1}}^i$ with $\omega_s = \sum_{i=1}^{I} \omega_s^i$ and define $X = \mathbb{R}_+ \times \prod_{s=1}^{S} [-\omega_s, \omega_s)$.

This definition of the consumption set X reflects our assumption that agents can plan negative consumption in some future states (but not at $t = 0$). Each agent is assumed to have a preference ordering over consumption plans defined by an additively separable, linear quadratic utility function

$$u^i(\boldsymbol{x}^i) = \alpha_0^i x_0^i - \frac{1}{2} \sum_{s=1}^{S} \rho_s (\alpha_1^i - x_s^i)^2, \quad i = 1, \ldots, I, \tag{1}$$

where the time 1 state probabilities are given by the ρ_s for $s = 1, \ldots, S$. We assume that all preference parameters $(\alpha_0^i, \alpha_1^i) \in \mathbb{R}_{++}^2$ are chosen such that for each agent $\alpha_1^i \mathbb{1} - \omega_{\mathbf{1}} \in \mathbb{R}_{++}^S$ where $\mathbb{1}$ is the S-dimensional vector consisting of components equal to 1. Such an assumption is used for instance in Nielsen [1989] to ensure existence of equilibrium with preferences allowing satiation. It ensures monotonicity of utility on those regions of the choice set X that correspond to a feasible allocation. A comprehensive discussion of assumptions dealing with satiation can be found in Sato [2010]. This specification of preferences assumes agents who care about the mean and the variance of their consumption plans. The advantage of this specific restriction is that it allows a characterization of equilibrium prices, portfolios and consumption.

To achieve a consumption profile optimally adapted to their risk preferences agents can trade $J + 1$ securities. These securities belong to two categories: First, there are J securities with payoff profile $y^j = (y_1^j, \ldots, y_S^j) \in \mathbb{R}^S$. In a richer model, where income is not given by endowments but generated in production, these payoffs could represent for instance the market value of the output of firms at $t = 1$. The $S \times J$ matrix of all security payoffs y^j is given by Y. Each consumer chooses a portfolio $z_e^i \in \mathbb{R}^J$ of these securities. Their prices are denoted by $q_e \in \mathbb{R}^J$. Together with the security payoffs Y they define the possible trades an agent can make by using these securities.

We do not allow agents to default by trading securities y^j. Thus, only trades are allowed which fulfill

$$\omega_{\mathbf{1}}^i + Y z_e^i \geq 0. \tag{2}$$

While there are many securities y^j there is only one security in the second class. It is a bond, which allows the agents to make loans. The bond promises to pay one unit of income in every state of the world. But since agents can take on debts by trading the bond and go bankrupt an agent who has invested in the bond has to take into account that the payoff profile of the bond is not $\mathbb{1} \in \mathbb{R}^S$ but only $r_{\mathbf{1}} \in [0,1]^S$. We define the positions of agent i long in the bond by z^i_{b+} and the positions short in the bond by z^i_{b-}. A portfolio is then a tuple $z^i = (z^i_{b+}, z^i_{b-}, z^i_e) \in Z$, where $Z = \mathbb{R}^2_+ \times \mathbb{R}^J$ is the cone of possible portfolio positions. The constraint given in equation (2) is required to hold additionally and will be added when we define the budget set of an agent.

Since in this paper we are not interested in the most general formulation of the bankruptcy model but rather in a formulation that has enough structure to allow for a description of equilibrium allocations and equilibrium credit risk we impose the following additional restriction on security payoffs. We assume that $\text{span}(Y) \cap \mathbb{R}^S_+ = 0$. This assumption ensures that no matter how the recovery rate $r_{\mathbf{1}}$ will materialize in equilibrium, as long as it is non-zero, it will never define a payoff profile that can be reproduced by a portfolio of securities y^j.[5]

We assume that the matrix $V = [\mathbb{1}|Y]$ has full column rank, that $J+1 < S$ (incomplete markets) and that $E(V^j) > 0$ for all securities $j = 1, \ldots, J$, where E denotes the expectation operator.

We define the matrix T by

$$T = \begin{bmatrix} -q_b & q_b & -q_e \\ r_{\mathbf{1}} & -\mathbb{1} & Y \end{bmatrix}$$

The set of feasible income transfers is given by

$$\mathcal{C} = \{\boldsymbol{\tau} \in \mathbb{R}^{S+1} | \boldsymbol{\tau} = Tz \quad z \in Z\}$$

Since Z is a cone this implies that $\mathcal{C}$ is a cone in $\mathbb{R}^{S+1}$. To see this note that for every $z \in Z$ we have also $\lambda z \in Z$. Therefore $Tz \in \mathcal{C}$ implies $T(\lambda z) = \lambda T \in \mathcal{C}$, thus $\mathcal{C}$ is a cone which is by definition in $\mathbb{R}^{S+1}$.

3.2. Bankruptcy

We want to allow for the case that in period $t = 0$ agents can take more debt than they can service in every state. Thus, an institutional arrangement for states in which agents go bankrupt and do not pay back their loans has to be introduced. *Bankruptcy* refers to a set of institutional arrangements specifying the reallocation of claims among economic agents. An agent is *bankrupt*, when the value of his debts exceeds the value of his assets. In the two period framework employed here this condition can be unambiguously defined.[6]

[5] This is a restrictive assumption that avoids tedious technicalities in the proof of existence of equilibrium.

[6] Such a formalization has been used in the literature in different versions by Modica et al. [1998], Sabarwal [2003], Araujo and Pascoa [2002]. It is also close to the framework of Eisenberg and Noe [2001], which shows how one can extend our bankruptcy rule to many loan instruments and non-anonymous bankruptcy in a pure balance sheet mechanics framework. A bankruptcy occurs if agents cannot repay their due liabilities. In contrast to Zame [1993] and Dubey et al. [2005], we do not allow agents to default on their loans. Agents will repay their debts as long as the value of their endowments and assets allows it. If liabilities exceed this value a bankruptcy occurs.

In case of a bankruptcy, all remaining assets of the debtor will be seized and distributed among the creditors. The remaining debt will be forgiven. In case of bankruptcy two institutional aspects are essential for the economic outcome: First, how will the remaining assets be distributed among the claim holders? Second, what kind of penalty will be imposed on the bankrupt agent for not paying back the contracted amount of debt?

Bankruptcy laws specify these rules. A penalty is necessary in order to avoid that consumers with monotone preferences have unbounded demand for debt. Penalties for bankruptcies usually consist in excluding bankrupt individuals, at least temporarily, from further credit and constraining their consumption to a minimum for some period. Usually these penalties depend also on the size of the losses which creditors suffer.

If a bankruptcy occurs existing claims of the asset holders can no longer be satisfied. We want to analyze our bankruptcy model as a model of perfect competition. Bankruptcy is anonymous. Anonymity of bankruptcy can be formalized by assuming that purchases and sales of the bond are implemented through some central clearing institution that spreads shortfalls on promised payments proportionally among creditors. The remaining assets of bankrupt agents $\omega_s^i + Y_s z_e^i < z_{b-}^i$ will be seized and distributed to the creditors. If repayments $\left(\sum_{i=1}^I z_{b-}^i \wedge (\omega_s^i + Y_s z_e^i)\right)$ fall short of aggregate promises $\sum_{i=1}^I z_{b-}^i$ in some state s then these claims will be reduced proportionally.[7] The *recovery rate* in state s, $r_s \in [0,1]$, is

$$r_s = \frac{\sum_{i=1}^I z_{b-}^i \wedge (\omega_s^i + Y_s z_e^i)}{\sum_{i=1}^I z_{b-}^i}.$$

If there is no activity in the bond market we define $r_s = 1$. When planning their consumption and investments consumers will take this *recovery rate* r_s into account as an expected parameter which *will be determined in equilibrium.*

Define the positive and negative parts of consumption by

$$\begin{aligned} \boldsymbol{x}^{i-} &:= \boldsymbol{x}^i \wedge 0 \quad \text{and} \\ \boldsymbol{x}^{i+} &:= \boldsymbol{x}^i \vee 0. \end{aligned}$$

We model the bankruptcy penalty by a *penalty function* which is strictly increasing in the value on which the debtor defaults in a bankruptcy. Notice that a consumption plan $\boldsymbol{x}^i = \boldsymbol{x}^{i+} + \boldsymbol{x}^{i-}$ can now become negative in some states. The consumer must be capable of evaluating $x_s^i < 0$. Her consumption set is no longer confined to the positive orthant. The quadratic utility function gives a *joint evaluation of consumption and bankruptcy penalties.*[8] The greater the extent of bankruptcy the lower the utility. Whether consumers will accept the risk of bankruptcy in some state will thus ultimately depend on their preferences.

[7] The operator $\wedge$ denotes the component wise minimum. The component wise maximum operator is defined as $\vee$. By Y_s we denote the s-th row of the matrix Y

[8] This observation has been made by Magill and Quinzii [2000].

3.3. Bankruptcy Equilibrium

The budget set of agent i is then given by:

$$\mathbb{B}^i(q, r_{\mathbf{1}}) = \left\{ \boldsymbol{x}^i \in X \mid \exists z^i \in Z : \boldsymbol{x}^i - \boldsymbol{\omega}^i = T(q, r_{\mathbf{1}}) z^i \text{ and } \omega^i_{\mathbf{1}} + Y z^i_e \geq 0 \right\},$$

where we partitioned the vector z^i into $z^i = (z^i_{b+}, z^i_{b-}, z^i_e)$. Note that the recovery rate on the bond in each state is taken as a parameter. Consumers are assumed to maximize their utility subject to this budget constraint. Notice that consumption x^i_s may become negative in some state s, indicating that the consumer is bankrupt in this state and receives a bankruptcy penalty corresponding to this negative consumption value.

Let $\boldsymbol{u} = (u^1, \ldots, u^I)$ and $\boldsymbol{\omega} = (\boldsymbol{\omega}^1, \ldots, \boldsymbol{\omega}^I)$. Let $V = [\mathbb{1}, Y]$ denote the exogenously given security payoffs. We denote by $\mathcal{E}(\boldsymbol{u}, \boldsymbol{\omega}, V)$ the corresponding economy.

Definition 1 (Equilibrium)**.** *A financial market equilibrium with bankruptcy of the economy $\mathcal{E}(\boldsymbol{u}, \boldsymbol{\omega}, V)$ is a tuple of consumption plans, portfolio choices, security prices and recovery rates $(\bar{\boldsymbol{x}}, \bar{z}_{b+}, \bar{z}_{b-}, \bar{z}_e, \bar{q}, \bar{r}_{\mathbf{1}}) \in (X)^I \times \mathbb{R}^{2I}_+ \times \mathbb{R}^{JI} \times \mathbb{R}^J \times [0,1]^S$ such that for all $i = 1, \ldots, I$*

(i) $\bar{\boldsymbol{x}}^i = \arg\max\{u^i(\boldsymbol{x}^i) \mid \boldsymbol{x}^i \in \mathbb{B}^i(\bar{q}, \bar{r}_{\mathbf{1}})\}$

(ii) $\sum_{i=1}^I \bar{z}^i_{b+} = \sum_{i=1}^I \bar{z}^i_{b-}$ *and* $\sum_{i=1}^I \bar{z}^i_e = 0$

(iii) $\sum_{i=1}^I \bar{\boldsymbol{x}}^{i+} = \sum_{i=1}^I \boldsymbol{\omega}^i$

In equilibrium feasibility of the consumption allocation is guaranteed by condition (iii). This does not preclude that consumers choose an amount of debt which leads to a negative value of the consumption plan in some state. Hence, $\bar{x}^i_s$ can be negative in some states, representing the bankruptcy penalty experienced by this consumer. Creditors hold rational expectations about the recovery rate in states where bankruptcy occurs. Only in this case security market clearing (*ii*) and good market clearing (*iii*) can be fulfilled simultaneously. Bankruptcy is factored into the asset price system $\bar{q}$. Note that the standard general equilibrium concept without bankruptcy is a special case of the financial market equilibrium with bankruptcy when $r_s = 1$ for all states s.

4. Bankruptcy Equilibrium: Results

4.1. Bankruptcy Equilibrium: Existence

We first show that our equilibrium concept is well defined. We show that under our assumptions on preferences, endowments and securities a bankruptcy equilibrium will always exist. From our proof of existence, which is detailed in the appendix, we can gain some insight into the economics of a bankruptcy equilibrium.

Proposition 1 (Existence of Bankruptcy Equilibrium)**.** *Let $\mathcal{E}(\boldsymbol{u}, \boldsymbol{\omega}, V)$ be a finance economy. If*

A1 : $u^i : X \to \mathbb{R}$ *is linear quadratic and* $X = \mathbb{R}_+ \times \mathbb{R}^S$, *A2* : $\boldsymbol{\omega}^i = (\omega_0^i, \omega_{\mathbf{1}}^i) \in X \cap \mathbb{R}_{++}^{S+1}$

A3 : *For each agent* $\alpha_1^i \mathbb{1} - \omega_{\mathbf{1}} \in \mathbb{R}_{++}^S$

A4 : *The matrix* $[\mathbb{1}|Y]$ *has column rank* $J+1$

A5 : $span(Y) \cap \mathbb{R}_+^S = 0$

then a bankruptcy equilibrium exists.

Proof: The proof is given in the appendix. □

Existence of a bankruptcy equilibrium has been proved in a more general setting by Sabarwal [2003] and for a slightly different version of the model by Modica et al. [1998] and Araujo and Pascoa [2002]. We would like to make a few remarks about our proof of existence of a bankruptcy equilibrium because we believe that this proof reveals some interesting features of the economics of a bankruptcy equilibrium that would be hidden, if we just invoked the general existence theorem by Sabarwal [2003].

Our proof first shows that for an arbitrary recovery rate $r_{\mathbf{1}}$ we can always find a security price vector $\bar{q}$ that clears the financial market. This does however not guarantee that we have a bankruptcy equilibrium according to our definition because this requires that $r_{\mathbf{1}}$ is chosen such that the good market clears at $t = 1$. This will be possible only if $r_{\mathbf{1}}$ corresponds to the actually realized equilibrium recovery rate, which must therefore be perfectly foreseen. An argument showing that we can always find such a corresponding $\bar{r}_{\mathbf{1}}$ completes the existence proof.

This proof reveals the special role of default expectations for the allocative consequences of credit risk. It might be possible that the prices in financial markets perfectly coordinate supply and demand for securities. Still if also inter-temporal consumption decisions should be perfectly coordinated by the market it has to be the case that the recovery rate is *perfectly* foreseen. Otherwise there will be rationing or slack resources in the economy. Note that a model of borrowing and lending with bankruptcy always has a trivial equilibrium where there is no trade in the bond. Indeed if everybody expects zero recovery from an investment in the bond then it is indeed optimal not to trade in the bond at all and the expectations are also consistent with these actions. This situation is somewhat reminiscent of general equilibrium models of money that always have equilibria where money has no value (see Gale [1982]). Zame [1993] and Dubey et al. [2005] make attempts to exclude these kinds of equilibria through some kind of refinement. We believe however that the possibility of such trivial equilibria in a context of bankruptcy and default are entirely natural and should indeed be part of the model. The role of default expectations is absolutely crucial in coordinating behavior. This feature of the bankruptcy model leads to interesting problems in a richer context with production where “wrong” expectations about the aggregate recovery rate would then lead – for instance – to unemployment or underinvestment.

4.2. Bankruptcy Equilibrium and No Arbitrage

In the following, we use the probability induced inner products

$$\langle x_{\mathbf{1}}, y_{\mathbf{1}} \rangle = \sum_{s=1}^{S} \rho_s x_s y_s \quad \forall\, x_{\mathbf{1}}, y_{\mathbf{1}} \in \mathbb{R}^S \text{ and} \tag{3}$$

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{s=0}^{S} \rho_s x_s y_s \quad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{S+1}, \tag{4}$$

where we define $\rho_0 := 1$.

Trading in financial markets allows agents to redistribute income across states of the world. In the bankruptcy model the set of feasible income transfers is a closed, convex and finitely generated cone given by

$$\mathcal{C} = \{\boldsymbol{\tau} \in \mathbb{R}^{S+1} | \boldsymbol{\tau} = Tz, \quad z \in Z\}.$$

Definition 2 (No Arbitrage)**.** *Absence of arbitrage in the financial market means that for a given recovery rate $r_{\mathbf{1}}$ security prices q have to be such that there does not exist any $z \in Z$ such that $Tz > 0$.*[9]

This condition can be characterized as follows:

Lemma 1. *Given $(q, r_{\mathbf{1}}, Y)$ there is no arbitrage if an only if there exists a vector $\boldsymbol{\pi} \in \mathbb{R}^{S+1}_{++}$ such that $\langle \boldsymbol{\pi}, \boldsymbol{\tau} \rangle \leq 0$ for all $\boldsymbol{\tau} \in \mathcal{C}$.*

Proof: The proof is given in the appendix □

In the following, we normalize $\boldsymbol{\pi}$ such that $\pi_0 = 1$ and write the date 1 components of $\boldsymbol{\pi}$ as $\pi_{\mathbf{1}}$. Let us define in addition to the matrix $V = [\mathbb{1}|Y]$ the matrix $W = [r_{\mathbf{1}}|Y]$. For a given recovery rate $r_{\mathbf{1}}$ it follows from Lemma 1 that the set of no arbitrage prices can be written as

$$Q(r_{\mathbf{1}}) = \{q \in \mathbb{R}^{J+1} \mid \pi_{\mathbf{1}} W \leq q \leq \pi_{\mathbf{1}} V \text{ for some } \pi_{\mathbf{1}} \in \mathbb{R}^S_{++}\}. \tag{5}$$

In the no bankruptcy case the no arbitrage condition allows the pricing of arbitrary income streams that can be generated from already existing securities by a linear pricing function. While with incomplete markets (normalized) state prices consistent with no arbitrage are not unique, the projection of $\pi_{\mathbf{1}}$ onto the space spanned by the existing security payoffs V is unique. Here, span(V) need not include $\mathbb{1}$. Denote this projection by $\pi_{\mathbf{1}}^V$ then the price of an arbitrary income stream $m \in \text{span}(V)$ in a bankruptcy equilibrium is given by

$$c(m) = \langle \pi_{\mathbf{1}}^V, m \rangle.$$

This pricing formula can be interpreted in statistical terms as

$$c(m) = E(\pi_{\mathbf{1}}^V) E(m) + \text{cov}(\pi_{\mathbf{1}}^V, m).$$

[9] $Tz > 0$ is a vector inequality. A vector $y \in \mathbb{R}^{S+1} > 0$ if and only if $y \in \mathbb{R}^{S+1}_+$ and $y \neq 0$

The vector $\pi_{\mathbf{1}}^V$ is in the span of traded securities and is sometimes also called the pricing security.

If, in addition, preferences are linear-quadratic for a financial market equilibrium the pricing result can be considerably sharpened. In this case, assuming that $\mathbb{1} \in \text{span}(V)$ and $\omega_{\mathbf{1}} \in \text{span}(V)$[10], we can write for any $m \in \text{span}(V)$ the pricing formula as

$$c(m) = \frac{E(m)}{1+r} - \frac{1}{\alpha_0}\text{cov}(\omega_{\mathbf{1}}, m),$$

where $\alpha_0 = \sum_{i=1}^{I} \alpha_0^i$, $\omega_{\mathbf{1}} = \sum_{i=1}^{I} \omega_{\mathbf{1}}^i$ and $1/(1+r)$ is the price of the risk-less income stream $\mathbb{1}$. This is the famous formula known from the consumption based capital asset pricing model, in which the aggregate endowment $\omega_{\mathbf{1}}$ is the market portfolio. In the following, we show how close to this result we can come in a bankruptcy equilibrium.

In the bankruptcy case the characterization of no arbitrage by the existence of positive state prices does not yield a linear pricing function. Let us denote the matrix $T_{\mathbf{1}} = (r_{\mathbf{1}}, -\mathbb{1}, Y)$. The marketed subset is then $\mathcal{C}_{\mathbf{1}} = \{m \in \mathbb{R}^S | m = T_{\mathbf{1}}\, z, \text{ and } z \in Z\}$ and the polar cone to the cone of net income transfers $\mathcal{C}$ is denoted by $\mathcal{C}^{\circ} = \{\boldsymbol{\pi} \in \mathbb{R}^{S+1} | \langle \boldsymbol{\pi}, \boldsymbol{\tau} \rangle \leq 0 \text{ for all } \boldsymbol{\tau} \in \mathcal{C}\}$.

Applying Theorem 3 in Elsinger and Summer [2001] we claim that for any $m \in \partial \mathcal{C}_{\mathbf{1}}$

$$c(m) = \max_{\boldsymbol{\pi} \in \mathcal{C}^{\circ} \cap \mathbb{R}^{S+1}_{++}} \left\langle \frac{\pi_{\mathbf{1}}}{\pi_0}, y \right\rangle.$$

It can be shown that the pricing functional (see Elsinger and Summer [2001] Corollary 5) c is sub-linear. It can be interpreted in statistical terms as

$$c(m) = \max_{\boldsymbol{\pi} \in \mathcal{C}^{\circ} \cap \mathbb{R}^{S+1}_{++}} \left\{ E\left(\frac{\pi_{\mathbf{1}}}{\pi_0}\right) E(m) + \text{cov}\left(\frac{\pi_{\mathbf{1}}}{\pi_0}, y\right) \right\}. \tag{6}$$

This is how far we can get with respect to the pricing of contingent claims in the bankruptcy model building on no arbitrage arguments only.

Note that this observation shows a close connection between security pricing in a bankruptcy equilibrium and the literature on security pricing with portfolio constraints (see Jouinni and Kallal [1995], Luttmer [1996] and Elsinger and Summer [2001]). This is of course due to the fact that we formalized the portfolio choice problem with bankruptcy as a choice from the convex set Z instead as from the linear space $\mathbb{R}^{J+1}$. As in the case with no bankruptcy we can sharpen the result in equation (6) if we make further assumptions about preferences. This is what we are going to do in the following section.

4.3. Prices and Allocations in Bankruptcy Equilibrium

Assume we have a bankruptcy equilibrium $(\bar{\boldsymbol{x}}, \bar{z}, \bar{q}, \bar{r}_{\mathbf{1}})$, where agents take optimal decisions by solving the problem

$$\max\{u^i(\boldsymbol{x}^i) \,|\, \boldsymbol{x}^i \in \mathbb{B}^i(\bar{q}, \bar{r}_{\mathbf{1}}\}. \tag{7}$$

[10] These conditions can be weakened. See Magill and Quinzii [1995], chapter 3 exercise 7.

The existence of a solution to the consumer optimization problem is characterized by the no arbitrage condition.

Lemma 2. *The consumer problem (7) has a solution if and only if the financial market is arbitrage-free.*

Proof: The proof is given in the appendix. See also [Magill and Quinzii, 1995, Theorem 9.3] for the case without bankruptcy. □

Since the linear quadratic utility function is differentiable and concave we have:

Lemma 3. *The consumer problem (7) has an interior solution* $(\bar{\boldsymbol{x}}^i)$ *if and only if* $\nabla u^i(\bar{\boldsymbol{x}}^i) \in \mathcal{C}^\circ$*, where the gradient* ∇ *is defined with respect to the inner product (4).*

Proof: The proof is given in the appendix. □

Using these results we can now give a description of a bankruptcy equilibrium in terms of equilibrium prices.

Proposition 2. *If* $(\bar{\boldsymbol{x}}, \bar{z}, \bar{q}, \bar{r}_{\mathbf{1}})$ *is a bankruptcy equilibrium of the economy* $\mathcal{E}(\boldsymbol{u}, \boldsymbol{\omega}, F)$ *with interior solutions then*

(i) there exist strictly positive constants a and b such that the pricing security $\bar{\gamma}$ *fulfills*

$$\bar{\gamma}_{\mathbf{1}} = a\,\mathbb{1} - b\,\tilde{\omega}_{\mathbf{1}},$$

(ii) the equilibrium market value of any income stream $m \in \mathcal{C}_{\mathbf{1}}$ *fulfills the weak inequality*

$$c(m) \geq \langle \bar{\gamma}_{\mathbf{1}}, m \rangle = E(\bar{\gamma}_{\mathbf{1}})E(m) - b\,cov(\tilde{\omega}_{\mathbf{1}}, m),$$

where $\tilde{\omega}_{\mathbf{1}} = \big(\omega_{\mathbf{1}} - (\mathbb{1} - r_{\mathbf{1}}) \sum_i \bar{z}^{i-}\big)$ *is the aggregate endowment* $\omega_{\mathbf{1}}$ *reduced by the aggregate losses from bankruptcy.*

Proof: The proof is given in the appendix. □

From Proposition 2 we see that security prices in a bankruptcy equilibrium look similar to security prices in a financial market equilibrium without bankruptcy and quadratic preferences. The most important change is that in a bankruptcy equilibrium the role taken by the aggregate endowment $\omega_{\mathbf{1}}$ is now replaced by the aggregate endowment corrected for the aggregate losses from bankruptcy $\tilde{\omega}_{\mathbf{1}}$. Since this quantity determines the pricing security, the prices of *all* securities whether or not they are affected by credit risk are influenced by the trading of a defaultable security.

Since in a bankruptcy equilibrium agents can't go short in the security promising $r_{\mathbf{1}}$ and also can't go long in the security $\mathbb{1}$ income streams m that can be generated from the existing securities can not anymore be valued by a linear function. However Proposition 2 shows that we can give valuation bounds for income streams that can be replicated, similar as in the literature on portfolio constraints (see Luttmer [1996]).

We can also characterize equilibrium allocations such that the relationship to two fund separation theorems characteristic for the CAPM can be seen. The structure of bankruptcy equilibrium requires that we characterize the consumption (default) plans of agents depending on whether they are long or short in the bond in a bankruptcy equilibrium. As in the case with pricing the role of the aggregate endowment $\omega_{\mathbf{1}}$ is now taken by the aggregate endowment corrected for aggregate credit losses $\tilde{\omega}_{\mathbf{1}}$. Since constraints on the possible bond positions (z_b^{i+}, z_b^{i-}) may bind for some agents, we get additional terms that depend on the Lagrangian multipliers of the respective constraints.

Proposition 3. *If in a bankruptcy equilibrium an agent i trades long in the bond, her equilibrium consumption plan is given by*

$$\bar{x}^i_{\mathbf{1}} = \omega^i_{\mathbf{1}} + P_{Y_{b+}} \left((\alpha^i_1 - \frac{\alpha^i_0}{\alpha_0}\alpha_1)\mathbb{1} - (\omega^i_{\mathbf{1}} - \frac{\alpha^i_0}{\alpha_0}\tilde{\omega}_{\mathbf{1}}) \right) - \sigma_{b+}\frac{\alpha^i_0}{\alpha_0} r_{\mathbf{1}e},$$

where $\sigma_{b+} := \sum_{i=1}^{I} \sigma^i_{b+}$ *is the sum of all agents' Lagrange multipliers corresponding to the constraints* $z^i_{b+} \geq 0$ *cf. A.3,* $P_{Y_{b+}}$ *is the projection on the span of the matrix* $(r_{\mathbf{1}}, Y)$ *and*

$$r_{\mathbf{1}e} := \frac{r_{\mathbf{1}} - P_Y(r_{\mathbf{1}})}{||r_{\mathbf{1}} - P_Y(r_{\mathbf{1}})||^2}.$$

If agent i trades short in the bond, her equilibrium allocation is given by

$$\bar{x}^i_{\mathbf{1}} = \omega^i_{\mathbf{1}} + P_{Y_{b-}} \left((\alpha^i_1 - \frac{\alpha^i_0}{\alpha_0}\alpha_1)\mathbb{1} - (\omega^i_{\mathbf{1}} - \frac{\alpha^i_0}{\alpha_0}\tilde{\omega}_{\mathbf{1}}) \right) - \sigma_{b-}\frac{\alpha^i_0}{\alpha_0} \mathbb{1}_e,$$

where $\sigma_{b-} := \sum_{i=1}^{I} \sigma^i_{b-}$ *is the sum of all agents' Lagrange multipliers corresponding to the constraints* $z^i_{b-} \geq 0$*, cf. A.3,* $P_{Y_{b+}}$ *is the projection on the span of the matrix* $(-\mathbb{1}, Y)$ *and*

$$\mathbb{1}_e := \frac{\mathbb{1} - P_Y(\mathbb{1})}{||\mathbb{1} - P_Y(\mathbb{1})||^2}.$$

If in equilibrium agent i does not trade in the bond, her equilibrium allocation is given by

$$\bar{x}^i_{\mathbf{1}} = \omega^i_{\mathbf{1}} + P_Y \left((\alpha^i_1 - \frac{\alpha^i_0}{\alpha_0}\alpha_1)\mathbb{1} - (\omega^i_{\mathbf{1}} - \frac{\alpha^i_0}{\alpha_0}\tilde{\omega}_{\mathbf{1}}) \right).$$

Proof: The proof is given in the appendix. □

From Proposition 3 we see that in the bankruptcy equilibrium consumption is characterized by an approximate linear risk sharing rule both for agents long and short in the bond with the aggregate endowment corrected for credit losses. At this point it might be helpful to clarify the relation between bankruptcy equilibrium and the CAPM.

5. Bankruptcy Equilibrium and the CAPM

A bankruptcy equilibrium characterized in Propositions 2 and 3 shows in many respects structural similarities with the famous capital asset pricing model (CAPM). There are also some important differences. We would now like to discuss bankruptcy equilibrium in the context of the CAPM.[11]

[11] Our discussion of the CAPM follows [Magill and Quinzii, 1995, Chap. 3 Exercise 5].

An economy with utility functions u^i and endowments $\boldsymbol{\omega}^i$ and a financial market structure given by a $S \times J$ matrix of securities A is said to fulfill the assumptions of the capital asset pricing model if preferences are mean variance, the span of A contains a riskless security $\mathbb{1}$ and for every agent i his date $t = 1$ endowment $\omega^i_{\mathbf{1}}$ lies in the span of A. Linear quadratic preferences - as we use in our paper - are a special case of mean variance preferences that allow a closed form description of equilibrium prices and allocations in the case without bankruptcy. In the usual CAPM story the economy is represented as a bond equity economy where agents have ownership shares in exogenously given production plans. These shares are their sole source of income. The spanning assumption is then automatically satisfied. Finally there are objective probabilities for the state of the world at $t = 1$. It is thus possible to describe income and consumption streams by their statistical properties.

In the case of linear-quadratic preferences, at an interior equilibrium, optimality conditions and the equilibrium condition of security market clearing imply that equilibrium prices can be written as

$$\bar{q} = \frac{1}{\alpha_0}\langle(\alpha_1 \mathbb{1} - \omega_{\mathbf{1}}), A\rangle \tag{8}$$

If the riskless security $\mathbb{1}$ is in the span of A this results in the CAPM pricing formula

$$q_j = \frac{E(A^j)}{1+r} - \frac{1}{\alpha_0}\text{cov}(\omega_{\mathbf{1}}, A^j) \tag{9}$$

This formula says that the price of a security depends on its expected payoff discounted at the risk-less rate and the covariance of the payoff stream with the aggregate endowment. The aggregate endowment is in this case the benchmark portfolio called the *market portfolio* in the terminology of the CAPM.

In the finance literature this pricing relation is usually formulated in returns per unit of income invested at date $t = 0$. If $q_j \neq 0$ the return of security A^j is defined as

$$R_{A^j} = \frac{A^j}{q_j} \tag{10}$$

The per unit return on the risk-less security $\mathbb{1}$ is defined as the risk-less return $R_{\mathbb{1}} = \bar{R} = (1+r)$. The excess return of security A^j is the difference $R_{A^j} - R_{\mathbb{1}}$ and its expected value is called the risk premium on A^j. The return on the market portfolio $R_{\omega_{\mathbf{1}}}$ is determined by the application of pricing equation (8) to the income stream $\omega_{\mathbf{1}}$.

This implies the famous CAPM formula (security market line formula) for equilibrium risk premiums given by

$$E(R_{A^j}) - \bar{R} = \frac{cov(R_{\omega_{\mathbf{1}}}, R_{A^j})}{var(R_{\omega_{\mathbf{1}}})}\left(E(R_{\omega_{\mathbf{1}}}) - \bar{R}\right) \tag{11}$$

which asserts that only systematic risk is priced in a CAPM equilibrium. As we see from this discussion this is an equivalent formulation for the pricing equation (8) formulated in returns instead of prices.

This pricing relation holds in the CAPM with linear quadratic preferences, even if the aggregate endowment is not in the span of A. The structure of the pricing relation is not affected when the span of A does not contain the risk-less income stream, though the formula changes slightly. In this case the role of the riskless income stream is replaced by the income stream that is nearest[12] in the span of A to the riskless income stream. The formulas get a slight bit messier because we loose the simple expression for the price of the risk-less income stream but the structure of the pricing formula and the security markets line formula remain basically intact.

The pricing relation in a bankruptcy equilibrium changes only slightly compared to a standard CAPM: First, since the net income transfers that are achievable by trading securities is a cone and not a linear space, the pricing function is now not linear any longer. Second the role of the aggregate endowment is now taken by the aggregate endowment corrected by the aggregate equilibrium losses from credit risk. Since in the case of linear quadratic preferences the CAPM pricing relation holds no matter whether or not the aggregate endowment is in the span of the existing securities this leaves the structure of the pricing relation unchanged. For more general mean variance preferences this structure would only carry over to the part of the credit loss corrected endowment that can be spanned by the existing securities (see Oh [1996]). Since credit risk affects systematic risk, the CAPM-like pricing structure in a bankruptcy equilibrium affects the price of *all* securities no matter whether they have a credit risk or not. In equilibrium credit risk affects risk premia across all securities traded in financial markets. Finally, unlike in the CAPM-equilibrium without bankruptcy the equilibrium prices of securities can not be written as functions of the exogenous parameters. This is because aggregate credit losses depend on portfolio decisions. Only if the equilibrium aggregate short position in the bond is known, $\tilde{\omega}_{\mathbf{1}}$ can be determined.

With respect to the equilibrium allocation things are a bit more involved. To see this consider again the no bankruptcy case with linear quadratic preferences first. In an equilibrium without bankruptcy where all agents trade with respect to a security matrix A it can be shown that the equilibrium allocation is given by

$$\bar{x}^i_{\mathbf{1}} = \omega^i_{\mathbf{1}} + P_A\left((\alpha^i_1 - \frac{\alpha^i_0}{\alpha_0}\alpha_1)\mathbb{1} - (\omega^i_{\mathbf{1}} - \frac{\alpha^i_0}{\alpha_0}\omega_{\mathbf{1}})\right) \tag{12}$$

where P_A denotes the orthogonal projection onto the span of A with respect to the inner product (3).

Comparing this allocation with the benchmark Arrow-Debreu (contingent market) equilibrium with normalized state prices $\pi = (1, \pi_1, \ldots, \pi_S)$ first order conditions imply that

$$\rho_s(\alpha^i_1 - x^i_s) = \pi_s \alpha^i_0 \tag{13}$$

Thus in equilibrium we must have

$$\rho_s \frac{(\alpha_1 - \omega_s)}{\alpha_0} = \pi_s \tag{14}$$

[12]in the norm induced by the probability inner product

Using the first order condition we can derive the vector equilibrium date 1 consumption in the contingent market equilibrium as

$$\bar{\bar{x}}^i_{\mathbf{1}} = \left(\alpha^i_1 - \alpha_1 \frac{\alpha^i_0}{\alpha_0}\right) \mathbb{1} + \frac{\alpha^i_0}{\alpha_0} \omega_{\mathbf{1}} \tag{15}$$

Going back to (12) we now see that under the assumptions of the CAPM with both $\mathbb{1}$ and $\omega_{\mathbf{1}}$ in the span of A, $P_A \mathbb{1} = \mathbb{1}$ and $P_A \omega_{\mathbf{1}} = \omega_{\mathbf{1}}$. Thus the CAPM assumptions imply that $\bar{x}^i_{\mathbf{1}} = \bar{\bar{x}}^i_{\mathbf{1}}$.

In this case we see why in the CAPM both the allocation fulfills a linear risk sharing rule (mutual fund theorem) and is at the same time pareto efficient even if the column rank of A is smaller than the number of states.

If the riskless security is not anymore in the span of A linear risk sharing stays in tact but the riskless security is replaced by the income stream in the span of A which is closest to the riskless income stream. Pareto optimality of risk sharing is lost. If also the aggregate endowment is not in the span the mutual fund property changes as well. In this case the endowment can be decomposed in a tradeable part $\omega^t_{\mathbf{1}}$ which lies in the span of A and a non-tradeable part $\omega^{nt}_{\mathbf{1}}$ which is in the orthogonal complement of the span of A. For the tradeable part of the endowment we still have a mutual fund theorem. Using (12) we see that in this case we have

$$\bar{x}^i_{\mathbf{1}} - \omega^{i^{nt}}_{\mathbf{1}} = \left(\alpha^i_1 - \frac{\alpha^i_0}{\alpha_0}\alpha_1\right) P_A \mathbb{1} + \frac{\alpha^i_0}{\alpha_0} \omega^t_{\mathbf{1}} \tag{16}$$

Pareto optimality is also not fulfilled in this case. We can however see from comparing (12) with (15) that the equilibrium net trade is the net trade that is closest in the span of A to the net trade that is realized in a Arrow Debreu (contingent market) equilibrium.

The efficiency properties as well as the form of the linear risk sharing rule that carries over to a bankruptcy equilibrium can be seen more clearly in the light of this discussion. The riskless income stream will not be generally available for all agents in a bankruptcy equilibrium. Furthermore, even if we assume that the aggregate endowment $\omega_{\mathbf{1}}$ is in the span of Y this does not guarantee that the aggregate endowment corrected for credit losses $\tilde{\omega}_{\mathbf{1}}$ is in the span of Y as well. Thus there is no a priori benchmark or market portfolio in a bankruptcy equilibrium. The market portfolio arises endogenously in equilibrium. Both of these features - no riskless income stream and no a priory benchmark portfolio in the span of available securities - imply that a bankruptcy equilibrium allocation cannot be pareto optimal.

A linear risk sharing rule carries over in the sense that the equilibrium consumption net of the non tradeable part of the individual endowment fulfills an approximate linear risk sharing rule for agents long, short and inactive in the bond. The additional factor that enters the equation comes from the fact that for some agents the constraints on the bond holdings may become binding. In the limit for bankruptcy risk going to zero ($r_{\mathbf{1}}$ going to $\mathbb{1}$) the bankruptcy equilibrium looks like a generalized CAPM equilibrium without specific assumptions about whether $\mathbb{1}$ or $\omega_{\mathbf{1}}$ is in the span of existing securities.

6. Endogenous Credit Risk

In this section we briefly discuss the standard analysis of portfolio credit risk (see McNeil et al. [2005]) and its application in capital regulation from the perspective of our model. In the standard analysis a portfolio credit risk model is a mapping from a set of individual credit instrument characteristics and market parameters into a loss distribution for the portfolio. In capital adequacy regulation this analysis is applied to the calibration of risk weights for different asset classes. The basic idea is that weights are calibrated in such a way that the capital contribution of each asset class is proportional to this asset class's contribution to the 99.9 % Value-at-Risk of the portfolio.[13] The capital calibrated in this way should then be able to support a portfolio with this risk profile in the sense that it is able to absorb all losses within the VaR range.

If credit risk is endogenous like in our model such an approach to risk weight calibration would not work. The reason is simply that in an equilibrium model credit risk analysis differs from a decision problem under risk, because the parameters of credit instruments depend on the equilibrium. Thus any imposition of risk weights that constrain individual behavior in some way will change the instrument parameters and thus the loss distribution. In such an environment risk weight calibration can soon become a fairly intractable problem.

To see this connection more clearly let us discuss the standard approach to credit risk modeling a bit more precisely and translate the concepts into the general equilibrium model. The standard approach usually considers a portfolio of exposures over a fixed time horizon. The exposures have a fixed recovery rate and the default of each creditor happens with a certain probability before the time horizon. To take into account default correlation, credit risk models usually assume that default probabilities depend on a common factor. For example, in the famous single factor model (see Schönbucher [2000]) asset values of creditors at the time horizon are dependent on a common factor and some idiosyncratic noise term. The default event occurs when asset values decline below a certain threshold. The credit risk parameters determine a loss distribution. Risk weights for asset classes are determined such that they are proportional to the contribution of assets in each class to some Value at Risk level in the loss distribution.

A portfolio of credit risk instruments occurs in our model at the level of the aggregate economy or at the level of the clearing house. Creditors are the agents who hold a short position in the bond. The clearing house holds at $t = 0$ a fixed claim with credit risk at $t = 1$ on each of these agents. Credits are financed by receipts from bond investors at $t = 0$. The clearing house has obligations to bond investors at $t = 1$. These obligations are fulfilled out of revenues from the creditors. The probability of default of a creditor i at time $t = 0$ in our case depends on the equilibrium because it is given by the probability that agent i plans a negative consumption at the time horizon $t = 1$. This probability

[13] To make this description more precise several technical qualifications have to be added. What is taken as the base for risk weight calibration under the Basel II advanced internal rating based approach is the asymptotic contribution of a specific risk to the 99.9 % VaR of the portfolio in a one-factor Gaussian threshold model with a specific asset correlation parameter. For a good reference, see Gordy [2001]

is the sum of all ρ_s where such a negative consumption would occur in equilibrium. The recovery rate for each creditor in a particular state of the world is the ratio of what he actually pays and what he promises and thus is also something determined in equilibrium. The pooled or average recovery rate in the portfolio is $r_{\mathbf{1}}$.

Assume that an expert committee is charged with prescribing a risk sensitive amount of equity against the portfolio of bonds the clearing house has bought from creditors at $t = 0$. The expert committee decides to achieve risk sensitivity by calibrating weights for each creditor such that the equity amount to be issued against the amount of loans granted to creditor i reflects creditor $i's$ contribution to the 99% Value at Risk of the loan portfolio. The committee has a sufficiently long history of data generated by bankruptcy equilibria which differ only by some idiosyncratic noise. Based on these data the committee comes up with estimated parameter values for the probability of default of each creditor, his exposure at default as well as the recovery rate. Assuming a credit risk model they would calculate a loss distribution for the clearing house and engineer an equity buffer that requires to issue w^i of equity against each unit of bond issued to creditor i. The contribution factor w^i is calibrated such that it corresponds to the marginal contribution to the 99% VaR under the estimated loss distribution based on the historical data.

The budget constraint of agents would change in two ways: Beyond the bond and the securities y^j we would now have a new security that promises in each state a share of the net worth from the difference between what is received from creditors and what is payed to bond investors. If this net worth becomes negative the clearing house is bankrupt and the losses are distributed proportionally to the bond holders. If this security is not in the span of existing securities the transfer space changes. The expectation about the recovery rate $r_{\mathbf{1}}$ now depends on the vector of risk weights, because these weights determine the amount of equity issued by the clearing house and thus the threshold beyond which bond holders have to share in credit losses. Both effects change relative prices of securities. As a consequence individual decisions as well as the aggregate consequences change and the parameters with which a credit loss distribution has been estimated are now different. In a world with endogenous credit risk, risk weight calibration becomes thus an extremely involved if not entirely intractable problem. But of course risk weighting can only work, if we can get the numbers on the weights (approximately) right. In a world where the parameters of credit risk are endogenous risk weight calibration according to the standard procedures is problematic if not flawed.

In the context of the model, we can also see more clearly that the question whether risk weight calibration for capital adequacy requirements is good or bad is not well posed (see Hellwig [2010]). As long as the precise objectives of the regulation remains unclear this question can not be answered in a meaningful way. We need a broader framework modeling the economics of credit risk to see more clearly what are the economic costs of credit risk, by which mechanisms precisely credit risk imposes costs on society at large and how it might be regulated. Hellwig [2010] points out that the buffer story of bank capital is only one of at least three conflicting versions about the mechanisms by which capital standards might influence the risk of a banking system.

Our model provides a general equilibrium framework in which questions of efficiency, the optimality of risk allocation from the viewpoint of society at large (welfare) can be addressed in principle. The seminal literature on credit risk in general equilibrium has indeed made the claim that the possibility of defaulting on loans may be welfare enhancing from an ex-ante viewpoint (see Dubey et al. [2005], Zame [1993]). The mechanism by which this welfare enhancement comes about is based on risk transfer options the traded security structure allows for in an incomplete markets setting. An extreme example would be as follows: If we had no financial instruments at all and then got a possibility to save and borrow, even if in some states a default and the dead-weight loss from punishment had to be incurred, the benefits from this possibility to reallocate income across time and states might be so large that they outweigh these costs. Examples given by Dubey et al. [2005] suggest that there might even be an optimal amount of credit risk for the economy as a whole. It is however not clear what would be the appropriate instrument to implement such an optimal amount. Our framework provides a starting point to look deeper into this question. This is something we would like to analyze in future research.

Thinking about credit risk in this way also raises the deeper question whether the way credit risk is modeled here captures in a useful way the salient features of the economic costs of default that occur in practice. Spanning arguments might in this respect be not the most important thing. Perhaps the length of the time span resources are left idle while the redistribution of claims and a future use of the assets has to be decided is an aspect much more crucial for the social costs of default than the spanning considerations highlighted in the model. In the model, even though defaults occur, the resolution of default is entirely without frictions. There is thus perhaps a need to formulate not only models that make the existing literature on credit risk in general equilibrium more tractable but also for models that highlight the frictions of resolving a bankruptcy as the key economic costs of credit risk.

7. Conclusions

This paper contributes to the literature about default in general equilibrium. It suggests a framework that allows us to use the central ideas of this literature while providing enough structure to describe equilibrium prices and allocations and going beyond abstract existence results and fully parametrized examples. We believe that using this approach we can mobilize the conceptual power of general equilibrium thinking for financial stability analysis. This power stems from the focus on the systemic aspects of economic interaction. This is highly needed in financial stability analysis where many of the tools and concepts used by regulators and policy makers look at risk analysis as a game against nature.

Taking the Capital Asset Pricing Model (CAPM) as a starting point, we develop a general equilibrium model of credit risk which embeds the CAPM as a special case. We describe the structure of equilibrium security prices and allocations in the presence of credit risk. While prices can be described in a very similar way than in the CAPM credit risk in the economy affects risk premia of all securities traded in the market. Allocations in a bankruptcy equilibrium retain some similarity with the mutual fund structure in the CAPM but the efficiency properties of a CAPM equilibrium are lost.

We apply our framework to an analysis of the prevailing thinking about credit risk in the regulatory debate. While in this debate credit risk is conceptually thought of as exogenous our model shows a world where all the parameters of credit risk – probability of default, exposure at default and recovery rate – are endogenous. We discuss the idea of engineering loss absorption buffers of equity for credit losses by designing risk weights for asset classes. We show that in a world of endogenous credit risk the approach to calibrating risk weights is problematic and potentially intractable. From a more general perspective our paper provides a framework in which the more fundamental question about what would in principle be the right approach to regulate credit risk can be addressed. Building on the results derived in this paper, we plan to tackle this question in subsequent research.

The policy debate on regulation still has very weak conceptual foundations. We hope that with this paper we can make a contribution to the attempts to improve this situation.

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A. Appendix

A.1. Proof of Lemma 1:

We can apply the same arguments as in the proof of Magill and Quinzii [1995] theorem 9.3. (iii). The only twist in our case is that the inequality $\boldsymbol{\pi}\boldsymbol{\tau} \leq 0$ cannot be turned into an equality because $\boldsymbol{\tau} \in \mathcal{C}$ does not imply that $-\boldsymbol{\tau} \in \mathcal{C}$. $\square$

A.2. Proof of Lemma 2:

Suppose first the consumer problem $\max\{u^i(\boldsymbol{x}^i) \,|\, \boldsymbol{x}^i \in \mathbb{B}^i(\bar{q}, \bar{r}_{\mathbf{1}}, \boldsymbol{\omega}^i)\}$ has a solution $\bar{\boldsymbol{x}}^i = \boldsymbol{\omega}^i + T\bar{z}^i$ such that $(\bar{x}^i_0, \bar{x}^i_{\mathbf{1}}) \in X$ but the financial market admits an arbitrage. Then there is a $z^i \in Z$ such that $Tz^i \geq 0$ and $Tz^i \neq 0$. Thus the consumption plan $\boldsymbol{x}^i := \boldsymbol{\omega}^i + T\bar{z}^i + Tz^i$ fulfills $\boldsymbol{x}^i \geq \bar{\boldsymbol{x}}^i$ and $\boldsymbol{x}^i \neq \bar{\boldsymbol{x}}^i$. By monotonicity of u^i on X we have $u^i(\boldsymbol{x}^i) > u^i(\bar{\boldsymbol{x}})^i$ but then $\bar{\boldsymbol{x}}^i$ can not be a solution to the consumer optimization problem.

Now assume that the financial market is arbitrage-free. Then, by Lemma 1 there is a $\boldsymbol{\pi} \in \mathcal{C}^\circ \cap \mathbb{R}^{S+1}_{++}$ with $\langle \boldsymbol{\pi}, \boldsymbol{\tau} \rangle \leq 0$. Define the corresponding contingent budget set $\mathbb{B}^i(\boldsymbol{\pi}) := \{\boldsymbol{x}^i \in X \mid \langle \boldsymbol{\pi}, (\boldsymbol{x}^i - \boldsymbol{\omega}^i) \rangle \leq 0\}$. Using that X is bounded from below, it can be shown (cf. [Magill and Quinzii, 1995, Proposition 7.3]) that $\mathbb{B}^i(\boldsymbol{\pi})$ is compact. Since $\mathbb{B}^i(\bar{q}, \bar{r}_{\mathbf{1}}, \boldsymbol{\omega}^i)$ is a closed subset of $\mathbb{B}^i(\boldsymbol{\pi})$ it is compact, too. Thus, the maximization of the continuous utility function u^i on $\mathbb{B}^i(\bar{q}, \bar{r}_{\mathbf{1}}, \boldsymbol{\omega}^i)$ has a solution. $\square$

A.3. Proof of Lemma 3:

We partition the matrix T and the portfolio vector z^i into long-bond, short-bond, and equity trades by $T = (T_{b+}, T_{b-}, T_e)$ and $z^i = (z^i_{b+}, z^i_{b-}, z^i_e)^T$, respectively. Using Lagrange multipliers $\boldsymbol{\pi}^i \in \mathbb{R}^{1+S}$, $\sigma^i_{b+} \geq 0$, and $\sigma^i_{b-} \geq 0$, The KKT conditions for the minimization of the Lagrange function

$$L^i(\boldsymbol{x}^i, z^i, \boldsymbol{\pi}^i, \sigma^i_{b+}, \sigma^i_{b-}) = -u^i(\boldsymbol{x}^i) + \langle \boldsymbol{\pi}^i, \boldsymbol{x}^i - \boldsymbol{\omega}^i - Tz^i \rangle - \sigma^i_{b+} z^i_{b+} - \sigma^i_{b-} z^i_{b-}$$

imply that

$$\begin{aligned} \langle \nabla u^i(\bar{\boldsymbol{x}}^i), T_e \rangle &= (0, \ldots, 0), \\ \langle \nabla u^i(\bar{\boldsymbol{x}}^i), T_{b+} \rangle &= -\sigma^i_{b+} \leq 0, \text{ and} \\ \langle \nabla u^i(\bar{\boldsymbol{x}}^i), T_{b-} \rangle &= -\sigma^i_{b-} \leq 0, \end{aligned}$$

from which follows that $\nabla u^i(\bar{\boldsymbol{x}}^i) \in \mathcal{C}^\circ$. Since the optimization problem is convex, the KKT conditions are also sufficient. $\square$

A.4. Proof of Proposition 2:

The gradient of the linear quadratic utility function is $\nabla u^i(\boldsymbol{x}^i) = (\alpha^i_0, \alpha^i_1 - x^i_{\mathbf{1}})^T$ and fulfills in the equilibrium allocation $\bar{\boldsymbol{x}}^i$ according to Lemma 3

$$\langle \nabla u^i(\bar{\boldsymbol{x}}^i), \boldsymbol{\tau} \rangle \leq 0 \quad \forall \boldsymbol{\tau} \in \mathcal{C}.$$

Summing up all agent's equilibrium gradients we define the vector

$$\boldsymbol{\gamma} := \sum_{i=1}^{I} \nabla u^i(\bar{\boldsymbol{x}}^i) = (\alpha_0, \alpha_1 \mathbb{1} - (\omega_{\mathbf{1}} - d_{\mathbf{1}}))^T,$$

where $d_{\mathbf{1}} = \sum_{i=1}^{I} d_{\mathbf{1}}^i$ is the aggregate credit loss of all agents. Still we have $\langle \boldsymbol{\gamma}, \boldsymbol{\tau} \rangle \leq 0 \quad \forall \boldsymbol{\tau} \in \mathcal{C}$. Since any trade $\boldsymbol{\tau} \in \mathcal{C}$ can be decomposed as $\boldsymbol{\tau} = (-c(m), m)$ we get for $\bar{\boldsymbol{\gamma}} := \frac{1}{\alpha_0}\boldsymbol{\gamma}$ that $\bar{\gamma}_{\mathbf{1}} = \frac{\alpha_1}{\alpha_0}\mathbb{1} - \frac{1}{\alpha_0}\tilde{\omega}_{\mathbf{1}}$ and $c(m) \geq \langle \bar{\gamma}_{\mathbf{1}}, m \rangle$, which proves the lemma. □

A.5. Proof of Proposition 3:

Suppose agent i goes long in the bond. Define her trading matrix by $T_{\text{long}} = \begin{pmatrix} -q_b & q_e \\ r_{\mathbf{1}} & Y \end{pmatrix}$. From the proofs of lemma A.3 and proposition A.4 we know that

$$\langle T_{\text{long}}^T, \nabla u^i(\bar{\boldsymbol{x}}^i) \rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and} \tag{17}$$

$$\langle T_{\text{long}}^T, \boldsymbol{\gamma} \rangle = \begin{pmatrix} -\sigma_{b+} \\ 0 \end{pmatrix}, \tag{18}$$

where $\nabla u^i(\boldsymbol{x}^i) = (\alpha_0^i, \alpha_1^i - x_{\mathbf{1}}^i)^T$ and $\boldsymbol{\gamma} = (\alpha_0, \alpha_1 \mathbb{1} - \tilde{\omega}_{\mathbf{1}})^T$. Divide equation (17) by α_0^i and equation (18) by α_0, subtract the equations and multiply the result by α_0^i again. With $\bar{\tau}_{\mathbf{1}}^i := \bar{x}_{\mathbf{1}}^i - \omega_{\mathbf{1}}^i$ this gives

$$\langle Y_{b+}^T, \bar{\tau}_{\mathbf{1}}^i \rangle = \langle Y_{b+}^T, (\alpha_1^i - \frac{\alpha_0^i}{\alpha_0}\alpha_1)\mathbb{1} - (\omega_{\mathbf{1}}^i - \frac{\alpha_0^i}{\alpha_0}\tilde{\omega}_{\mathbf{1}}) \rangle - \frac{\alpha_0^i}{\alpha_0} \begin{pmatrix} \sigma_{b+} \\ 0 \end{pmatrix}.$$

As $\text{span}(Y) \cap \mathbb{R}_+^S = 0$ (i. e. assumption $A5$) holds, we can write

$$\begin{pmatrix} \sigma_{b+} \\ 0 \end{pmatrix} = \langle Y_{b+}^T, \sigma_{b+} \frac{r_{\mathbf{1}} - P_Y(r_{\mathbf{1}})}{||r_{\mathbf{1}} - P_Y(r_{\mathbf{1}})||^2} \rangle = \langle Y_{b+}^T, \sigma_{b+} r_{\mathbf{1}e} \rangle.$$

Now, since $\langle Y_{b+}^T, v_{\mathbf{1}} \rangle = \langle Y_{b+}^T, P_{Y_{b+}}(v_{\mathbf{1}}) \rangle$ for any vector $v_{\mathbf{1}} \in \mathbb{R}^S$, it follows that

$$\bar{\tau}_{\mathbf{1}}^i = P_{Y_{b+}} \left((\alpha_1^i - \frac{\alpha_0^i}{\alpha_0}\alpha_1)\mathbb{1} - (\omega_{\mathbf{1}}^i - \frac{\alpha_0^i}{\alpha_0}\tilde{\omega}_{\mathbf{1}}) - \sigma_{b+}\frac{\alpha_0^i}{\alpha_0} r_{\mathbf{1}e} \right).$$

Finally, since $r_{\mathbf{1}} - P_Y(r_{\mathbf{1}}) \in \text{span}(Y_{b+})$, the result follows.

The results for agents going short and for agents that do not trade in the bond are proved similarly. □

A.6. Proof of Proposition1:

Before we proof the proposition we have to establish a number of auxiliary results. We first always assume that $r_{\mathbf{1}} \neq 0$ and deal with the case $r_{\mathbf{1}} = 0$ separately.

Define the matrices $W = [r_{\mathbf{1}}|Y]$ and $V = [\mathbb{1}|Y]$. For any $r_{\mathbf{1}} \in [0,1]^S$ define the set of arbitrage-free security prices at $r_{\mathbf{1}}$ by

$$Q(r_{\mathbf{1}}) = \{q \in \mathbb{R}^{J+1}_{++} \mid \exists \pi \in \mathbb{R}^S_{++}\ \pi W \leq q \leq \pi V\}$$

By Lemma (2) the consumer problem is well defined for any $q \in Q(r_{\mathbf{1}})$.

We begin with the characterization of the optimal decisions of agents.

Lemma 4. *When $q \in Q(r_{\mathbf{1}})$ the decision problem of each agent has a unique solution $(\bar{\boldsymbol{x}}^i, \bar{z}^i)$ that satisfies $\bar{z}^i_{b+}\bar{z}^i_{b-} = 0$, i.e. at each profile of optimal decisions the set of agents is partitioned in agents long, agents short and agents inactive in the bond market.*

Proof: The fact that at an optimal decision an agent is either long, short or inactive in the bond market can be shown directly from the KKT conditions which we have derived in Lemma (A.3). If $z^i_{b+} > 0$ and $z^i_{b-} > 0$ then $\sigma^i_{b+} = 0$ and $\sigma^i_{b-} = 0$. But then $\langle \nabla u^i(\bar{\boldsymbol{x}}^i), T_{b+}\rangle = \langle \nabla u^i(\bar{\boldsymbol{x}}^i), T_{b-}\rangle$. This is possible only if there is no bankruptcy in equilibrium. In terms of the model this means $r_{\mathbf{1}} = \mathbb{1}$. If on the other hand $z^i_{b+} > 0$ and $z^i_{b-} = 0$ or $z^i_{b+} = 0$ and $z^i_{b-} > 0$ then we have either

$$\begin{aligned} \langle \nabla u^i(\bar{\boldsymbol{x}}^i), T_{b+}\rangle &= 0 \text{ and} \\ \langle \nabla u^i(\bar{\boldsymbol{x}}^i), T_{b-}\rangle &\leq 0 \end{aligned}$$

or

$$\begin{aligned} \langle \nabla u^i(\bar{\boldsymbol{x}}^i), T_{b+}\rangle &\leq 0 \text{ and} \\ \langle \nabla u^i(\bar{\boldsymbol{x}}^i), T_{b-}\rangle &= 0 \end{aligned}$$

Both systems are always compatible because $r_{\mathbf{1}} \in [0,1]^S$. If $z^i_{b+} = 0$ and $z^i_{b-} = 0$ then both $\sigma^i_{b+} > 0$ and $\sigma^i_{b-} > 0$. In this case $z^i_{b+} = 0$ and $z^i_{b-} = 0$ is the optimal decision because any investment costs more than the net present value of its future payments or any short position creates less value today than has to be paid back tomorrow. Therefore any optimal solution to the consumer problem has the property that a consumer is either long or short or inactive in the bond market but he is never simultaneously long and short in the bond.

By assumption [A1] the optimal consumption choice $\bar{\boldsymbol{x}}^i$ is unique. By assumptions [A4] and [A5] both the matrices $V = [\mathbb{1}\ Y]$ and $W = [r_{\mathbf{1}}\ Y]$ have full column rank. As a consequence the mappings $z^i \mapsto Vz^i + \omega^i_{\mathbf{1}}$ and $z^i \mapsto Wz^i + \omega^i_{\mathbf{1}}$ are injective. The asset portfolios are thus uniquely determined by the solution to the equations

$$\begin{aligned} Wz^i(q;r_{\mathbf{1}}) &= x^i_{\mathbf{1}}(q;r_{\mathbf{1}}) - \omega^i_{\mathbf{1}} \\ Vz^i(q;r_{\mathbf{1}}) &= x^i_{\mathbf{1}}(q;r_{\mathbf{1}}) - \omega^i_{\mathbf{1}} \end{aligned}$$

□

We define the budget correspondence $\mathbb{B}^i \mapsto \mathbb{R}^{S+1}$ by

$$\mathbb{B}^i(q;r_{\mathbf{1}}) = \{\boldsymbol{x}^i \in \mathbb{R}^{S+1} \mid \exists z^i \in Z \quad \boldsymbol{x}^i - \boldsymbol{\omega}^i \leq Tz^i \quad Tz^i_e + \boldsymbol{\omega}^i \geq 0\}$$

Lemma 5. *For any $r_{\mathbf{1}} \in [0,1]^S$, the budget correspondence is continuous and $x^i(q;r_{\mathbf{1}})$ and $z^i(q;r_{\mathbf{1}})$ are continuous functions of q.*

Proof: By Lemma (2) if $q \in Q(r_{\mathbf{1}})$ then $\mathbb{B}^i(q;r_{\mathbf{1}})$ is compact. Let q^n be any sequence in $Q(r_{\mathbf{1}})$ converging to $\bar{q}$ and $\boldsymbol{x}_n^i(q^n;r_{\mathbf{1}}) \in \mathbb{B}^i(q;r_{\mathbf{1}})$. Then by definition of the budget correspondence there is a $z_n^i(q^n;r_{\mathbf{1}})$ such that $\boldsymbol{x}_n^i(q^n;r_{\mathbf{1}}) - \boldsymbol{\omega}^i \leq Tz_n^i$ and $Tz_{en}^i + \boldsymbol{\omega}^i \geq 0$. Define $t^n = Tz_n^i$. t_n must be bounded. Assume it was unbounded then $\lim_{n\to\infty} \|t^n\| = +\infty$. Multiply the budget constraint with $1/\|t^n\|$. Since the budget is bounded from below the right hand side converges to a vector v with $\|v\| = 1$ while the left hand side converges to 0. This is a contradiction to the no arbitrage condition which requires that $\mathcal{C} \cap \mathbb{R}_{++}^{S+1} = \emptyset$. Thus t^n is a bounded sequence and thus has a convergent subsequence. By Hildenbrand and Kirmann [1991] Theorem AIII.1 the budget correspondence is uhc. The correspondence is also lhc. Since $\boldsymbol{\omega}^i \in X \cap R_{++}^{S+1}$ the correspondence $b^i(q;r_{\mathbf{1}}) = \{\boldsymbol{x}^i \in \mathbb{R}^{S+1} \mid \exists z^i \in Z \quad \boldsymbol{x}^i - \boldsymbol{\omega}^i < Tz^i \quad Tz_e^i + \boldsymbol{\omega}^i > 0\}$ is non-empty. Let $\boldsymbol{x}_n^i$ and z_n^i denote sequences with $\boldsymbol{x}_n^i \to \bar{\boldsymbol{x}}$ and $z_n^i \to \bar{z}$, where $\bar{\boldsymbol{x}} \in \mathbb{B}^i(q;r_{\mathbf{1}})$ for some z. Then for every q^n such that $q^n \to \bar{q}$ for n large enough $\boldsymbol{x}_n^i - \boldsymbol{\omega}^i < Tz_n^i$ and $Tz_{en}^i + \boldsymbol{\omega}^i > 0$. Thus $\boldsymbol{x}_n^i \in b^i(q;r_{\mathbf{1}})$. Therefore $b^i(q;r_{\mathbf{1}})$ is lhc at $\bar{q}$. By [A4] and [A5] $\mathbb{B}^i(q;r_{\mathbf{1}}) = \text{cl } b^i(q;r_{\mathbf{1}})$. Since the closure of a lhc correspondence is also lhc, the budget correspondence is lhc. Since the budget correspondence is uhc and lhc it is continuous. Since $\boldsymbol{x}^i(q;r_{\mathbf{1}}) = \arg\max\ u^i(\boldsymbol{x}^i)$ on $\mathbb{B}^i(q;r_{\mathbf{1}})$ it follows from the Berge Maximum Theorem (see for instance Border [1985]) that $\boldsymbol{x}^i(q;r_{\mathbf{1}})$ is continuous on $Q(r_{\mathbf{1}})$. Since $z^i(q;r_{\mathbf{1}})$ is unique by Lemma (4) $z^i(q;r_{\mathbf{1}})$ is also continuous on $Q(r_{\mathbf{1}})$. □

Lemma 6. *Let $z_0^i(q;r_{\mathbf{1}}) = x_0^i(q;r_{\mathbf{1}}) - \omega_0^i$. Define the individual excess demand functions by $f^i(q;r_{\mathbf{1}}) = \big(z_0^i(q;r_{\mathbf{1}}), z^i(q;r_{\mathbf{1}})\big)$. $f^i(q;r_{\mathbf{1}})$ is*

(i) continuous on $Q(r_{\mathbf{1}})$

(ii) homogeneous of degree 0: $f^i(\lambda q;r_{\mathbf{1}}) = f^i(q;r_{\mathbf{1}})$ for all $\lambda > 0$ for all $q > 0$.

(iii) bounded below: Define $A = [r_{\mathbf{1}}| - \mathbb{1}|Y]$. $\big(z_0^i(q;r_{\mathbf{1}}), Az^i(q;r_{\mathbf{1}})\big) \geq (-\omega_0^i, -(\omega_{\mathbf{1}}^i + \omega_{\mathbf{1}}))$ for all $q \in Q(r_{\mathbf{1}})$.

(iv) fulfills Walras law $qf^i(q;r_{\mathbf{1}}) = 0$ for all $q \in Q(r_{\mathbf{1}})$.

(v) If q^n is a sequence in $Q(r_{\mathbf{1}})$ such that $q^n \to \bar{q} \in \partial Q(r_{\mathbf{1}})$ or diverging, i.e. $\|q^n\| \to \infty$ then $(z_0^i(q;r_{\mathbf{1}}), z^i(q;r_{\mathbf{1}})) \to \big(\bar{z}_0^i, \bar{z}^i\big)$ such that $\boldsymbol{\omega}^i + Tz(\bar{q}, r_{\mathbf{1}}) \in \partial X$ or $\|z_0^i(q;r_{\mathbf{1}}), z^i(q;r_{\mathbf{1}})\| \to \infty$.

Proof:

(i) The continuity of the individual excess demand function follows directly from Lemma (5).

(ii) The budget set of agent i is then given by:

$$\mathbb{B}^i(q, r_{\mathbf{1}}) = \left\{ (x_0^i, x_{\mathbf{1}}^i) \in X \;\middle|\; \begin{array}{lcl} \exists (z_{b+}^i, z_{b-}^i, z_e^i) \in Z \text{ such that} & & \\ q_0(x_0^i - \omega_0^i) & = & -q_b z_{b+}^i + q_b z_{b-}^i - q_e z_e^i, \\ x_{\mathbf{1}} - \omega_{\mathbf{1}} & = & r_{\mathbf{1}} z_{b+}^i - \mathbb{1} z_{b-}^i + Y z_e^i, \text{ and} \\ \omega_{\mathbf{1}} + Y z_e^i & \geq & 0 \end{array} \right\}$$

Note that we have so far used the price normalization $q_0 = 1$, which we could work with precisely because of homogeneity. Clearly scaling (q_0, q_b, q_e) with $\lambda > 0$ leaves the budget set and thus demand functions for consumption and securities unchanged.

(iii) The lower bounds results in a straightforward way from the definition of X because $x_0^i \in \mathbb{R}_+$ and $x_{\mathbf{1}}^i \in \prod_{i=1}^I [-\omega_{\mathbf{1}}, \omega_{\mathbf{1}})$.

(iv) Follows from the definition of the budget set and the fact that u^i is monotone on X.

(v) Suppose this claim is not true and $f^i(q^n; r_{\mathbf{1}}) \to (\bar{z}_0^i, \bar{z}^i)$ such that $\boldsymbol{\omega}^i + T\bar{z}^i \in \text{int}X$ where $\bar{z}_0^i = \bar{x}_0^i - \omega_0^i$ for some positive $\bar{x}_0^i$.

We consider first the case where $q^n \to \bar{q} \in \partial Q(r_{\mathbf{1}})$. We show that $(\bar{z}_0^i, \bar{z}^i)$ maximizes utility at $\bar{q}$. Since $z^i(q^n; r_{\mathbf{1}})$ is an optimal choice, $u^i(\boldsymbol{\omega}^i + Tz^i(q^n; r_{\mathbf{1}})) \geq u^i(\boldsymbol{\omega}^i)$. Assume that $(\bar{z}_0^i, \bar{z}^i)$ is not optimal at $\bar{q}$. Then there exists a $(\tilde{z}_0^i, \tilde{z}^i)$ with $u^i(\boldsymbol{\omega}^i + T\tilde{z}^i(\bar{q}; r_{\mathbf{1}}) > u^i(\boldsymbol{\omega}^i + T\bar{z}^i(\bar{q}; r_{\mathbf{1}}))$. Since u^i and $z^i(q; r_{\mathbf{1}})$ are continuous there exists a $N \in \mathbb{N}$ such that for all $n > N$ there is a $z^i(q^n, r_{\mathbf{1}})$ with $u^i(\omega^i + Tz^i(q^n, r_{\mathbf{1}})) > u^i(\omega^i + T\bar{z}^i)$ such that $z^i(q^n, r_{\mathbf{1}}))$ is affordable at q^n. By continuity then $u^i(\omega^i + T\tilde{z}^i(q^n, r_{\mathbf{1}})) > u^i(\omega^i + Tz^i(q^n, r_{\mathbf{1}}))$ for n sufficiently large, contradicting the optimality of $z^i(q^n, r_{\mathbf{1}}))$. Thus $(\bar{z}_0^i, \bar{z}^i)$ must be optimal at $\bar{q}$ but since $\bar{q}$ is an arbitrage price it is not possible that $(\bar{z}_0^i, \bar{z}^i)$ is an interior solution by Lemma (2). Thus the statement can only be false if $\|q^n\| \to \infty$

Let us now consider the case $\|q^n\| \to \infty$. We have so far used the price normalisation $q_0 = 1$. By homogeneity we now re-normalize prices to $q_0 = 1/\|q^n\|$ for date 0 consumption and $q^n/\|q^n\|$ for securities. Suppose that $q^n/\|q^n\| \to \bar{q}$ and $\|z_0^i(q^n/\|q^n\|; r_{\mathbf{1}}), z^i(q^n/\|q^n\|; r_{\mathbf{1}})\|$ converges to some $(\bar{z}_0^i, \bar{z}^i)$ and $\bar{z}^i$ is a portfolio that induces a utility maximizing consumption bundle at $(\bar{q}_0, \bar{q})$. If not then there exists a $(\tilde{z}_0^i, \tilde{z}^i)$ with $u^i(\boldsymbol{\omega}^i + T\tilde{z}^i(\bar{q}; r_{\mathbf{1}})) > u^i(\boldsymbol{\omega}^i + T\bar{z}^i(\bar{q}; r_{\mathbf{1}}))$. Since u^i and $z^i(q; r_{\mathbf{1}})$ are continuous there exists a $N \in \mathbb{N}$ such that for all $n > N$ there is a $z^i(q^n, r_{\mathbf{1}})$ with $u^i(\omega^i + Tz^i(q^n, r_{\mathbf{1}})) > u^i(\omega^i + T\bar{z}^i)$ such that $z^i(q^n, r_{\mathbf{1}}))$ is affordable at at prices $(1/\|q^n\|, q^n/\|q^n\|)$. By continuity then $u^i(\omega^i + T\tilde{z}^i(q^n, r_{\mathbf{1}})) > u^i(\omega^i + Tz^i(q^n, r_{\mathbf{1}}))$ for n sufficiently large, contradicting the optimality of $z^i(q^n, r_{\mathbf{1}}))$. Consequently $\bar{z}^i$ induces a utility maximizing consumption bundle at prices $(\bar{q}_0, \bar{q})$. This leads to a contradiction, since we have assumed that $\|q^n\| \to \infty$ it must be the case that $1/\|q^n\| \to 0$ so $\bar{q}_0 = 0$ but at this price agents can choose unbounded date 0 consumption.

□

We now define $G(q; r_{\mathbf{1}}) = \sum_{i=1}^I f^i(q; r_{\mathbf{1}})$ as the security aggregate excess demand function. We want to show the existence of a q^* such that $G(q^*; r_{\mathbf{1}}) = 0$. We rely on a result due to Grandmont [1977] derived from Debreu [1956]. See also Border [1985] and Magill and Quinzii [1995].

Lemma 7. *Let $\bar{Q}(r_{\mathbf{1}}) \in \mathbb{R}^{J+1}$ be a closed convex cone which is not a linear space. Let $Q(r_{\mathbf{1}})$ denote the interior of $\bar{Q}(r_{\mathbf{1}})$ and let $\hat{Q}(r_{\mathbf{1}}) = \{q \in Q(r_{\mathbf{1}}) \mid \|q\| = 1\}$. If $G : \hat{Q}(r_{\mathbf{1}}) \to \mathbb{R}^{J+1}$ is a continuous function which satisfies $qG(q; r_{\mathbf{1}}) = 0$ for all $q \in \hat{Q}(r_{\mathbf{1}})$ and the boundary property given below holds then there is a $q^* \in \hat{Q}(r_{\mathbf{1}})$ such that $G(q^*; r_{\mathbf{1}}) = 0$. The boundary property is: If $q^n \to \bar{q}$ with $\bar{q} \in \partial\hat{Q}(r_{\mathbf{1}})$ and $q^n \in \hat{Q}(r_{\mathbf{1}})$ for all $n \in \mathbb{N}$, there exists a $\hat{q} \in \hat{Q}(r_{\mathbf{1}})$ such that for n sufficiently large, $\hat{q}G(q^n; r_{\mathbf{1}}) > 0$.*

Proof: For a proof of this results see Border [1985] chapter 18. □

Proof of Proposition 1 Define the set

$$\bar{Q}(r_{\mathbf{1}}) = \{q \in \mathbb{R}^{J+1}_{+} \mid \exists \pi \in \mathbb{R}^{S+1}_{+} \ \pi W \leq q \leq \pi V\}$$

By [A4] and [A5] this is the closure of the set $Q(r_{\mathbf{1}})$. Since $\bar{Q}(r_{\mathbf{1}})$ is also a convex cone, the set $\bar{Q}(r_{\mathbf{1}})$ is a closed and convex cone. Lemma (7) can therefore be applied to $G(q; r_{\mathbf{1}})$. Since the individual excess demand functions are homogeneous of degree zero the aggregate excess demand function $G(q; r_{\mathbf{1}})$ is homogeneous of degree zero we can renormalize prices and and consider the restriction to $\hat{Q}(r_{\mathbf{1}})$. This function fulfills Walras law, because the individual excess demand function fulfill Walras' law. It therefore remains to check the boundary property of Lemma (7). Let $q^n \to \bar{q}$ with $\bar{q} \in \partial\hat{Q}(r_{\mathbf{1}})$ and $q^n \in \hat{Q}(r_{\mathbf{1}})$ for all $n \in \mathbb{N}$. Consider $\hat{q}\ G(q^n; r_{\mathbf{1}})$ for some $\hat{q} \in \hat{Q}(r_{\mathbf{1}})$. In this case we know from the boundary behavior of individual excess demand functions that $\lim_{n\to\infty} G(q^n; r_{\mathbf{1}}) \to (\bar{z}^i_0, \bar{z}^i)$ inducing a consumption plan in the boundary of the feasible set or $\lim_{n\to\infty} \|G(q^n; r_{\mathbf{1}})\| \to \infty$. Define the matrix $A = [r_{\mathbf{1}}, -\mathbb{1}, Y]$. Since $\hat{q} \in \hat{Q}$, we have the inequality $\hat{q}\ G(q^n; r_{\mathbf{1}}) \geq \hat{\pi} AG(q^n; r_{\mathbf{1}})$ and $\hat{\pi} \in \mathbb{R}^S_{++}$. Since $AG(q^n; r_{\mathbf{1}})$ has at least one positive element $\hat{\pi}$ can always be chosen such that for n sufficiently large $\hat{\pi} AG(q^n; r_{\mathbf{1}}) > 0$ and thus $\hat{q}\ G(q^n; r_{\mathbf{1}}) > 0$. In the other case $\lim_{n\to\infty} \|AG(q^n; r_{\mathbf{1}})\| \to \infty$. Since $AG(q^n; r_{\mathbf{1}})$ is bounded below there must exist some N such that $\hat{q}G(q^n; r_{\mathbf{1}}) > 0$ for all $n > N$. Thus in both cases by Lemma (7), there exists a $q^* \in \hat{Q}(r_{\mathbf{1}})$ such that $G(q^*; r_{\mathbf{1}}) = 0$.

Now consider the special case $r_{\mathbf{1}} = 0$ next. In this case every agent expects the bond to pay zero and given these expectations no trade in the bond is indeed consistent with an equilibrium in which only securities Y^j are traded. But this is just a standard CAPM equilibrium without a riskless asset which exists by standard results in the literature (see for instance Dana [1996]). So indeed for every $r_{\mathbf{1}} \in [0,1]^S$ there is a $q^* \in \hat{Q}(r_{\mathbf{1}})$ such that $G(q^*; r_{\mathbf{1}}) = 0$.

Note, however that this is not yet a bankruptcy equilibrium, since $r_{\mathbf{1}} \in [0,1]^S$ has been chosen arbitrary. For arbitrary $r_{\mathbf{1}}$ the equilibrium condition $\sum_{i=1}^I \bar{\boldsymbol{x}}^{i+} = \sum_{i=1}^I \boldsymbol{\omega}^i$ will not be fulfilled.

For any $q \in Q(r_{\mathbf{1}})$ define the correspondence $R(r_{\mathbf{1}}; q) = (r(r_1; q), \ldots, r(r_s; q)) : [0,1]^S \mapsto [0,1]^S$ by

$$r(r_s; q) = \begin{cases} \frac{\sum_{i=1}^I z^i_{b-}(q; r_{\mathbf{1}}) \wedge (\omega^i_s + Y_s z^i_e(q; r_{\mathbf{1}}))}{\sum_{i=1}^I z^i_{b+}(q; r_{\mathbf{1}})} & \text{if } \sum_{i=1}^I z^i_{b+}(q; r_{\mathbf{1}}) > 0 \\ [0,1] & \text{otherwise} \end{cases}$$

Note that in a bankruptcy equilibrium the equilibrium condition $\sum_{i=1}^I \bar{\boldsymbol{x}}^{i+} = \sum_{i=1}^I \boldsymbol{\omega}^i$ and the requirement that $r_{\mathbf{1}}$ is a fixed point of $R(r_{\mathbf{1}}; q)$, i.e. $R(r_{\mathbf{1}}; q) = r_{\mathbf{1}}$, are equivalent.

To see this, consider the case $\sum_{i=1}^I z_{b+}^i(q;r_{\mathbf{1}}) > 0$ first. For any $s \in S$ summing

$$x_s^i = \omega_s^i + r_s z_{b+}^i - z_{b-}^i + Y_s z_e^i$$

over all agents yields

$$\begin{aligned}
\sum_{i=1}^I (x_s^i - \omega_s^i) &= \sum_{i=1}^I \left(r_s z_{b+}^i - z_{b-}^i + Y_s z_e^i\right) \\
&\Longleftrightarrow \\
\sum_{i=1}^I \left((x_s^i \vee 0) + (x_s^i \wedge 0) - \omega_s^i\right) &= \sum_{i=1}^I \left(r_s z_{b+}^i - z_{b-}^i + Y_s z_e^i\right) \\
&\Longleftrightarrow \\
\sum_{i=1}^I \left((x_s^i \vee 0) - \omega_s^i\right) &= \sum_{i=1}^I \left(r_s z_{b+}^i - z_{b-}^i + Y_s z_e^i - (x_s^i \wedge 0)\right)
\end{aligned}$$

In a bankruptcy equilibrium $\sum_{i=1}^I \left((x_s^i \vee 0) - \omega_s^i\right) = 0$. By the equivalence derived above this can be true if an only if $\sum_{i=1}^I \left(r_s z_{b+}^i - z_{b-}^i - (x_s^i \wedge 0)\right) = 0$, since $Y_s \sum_{i=1}^I z_e^i = 0$. Now this equation can hold if and only if

$$\begin{aligned}
r_s &= \frac{\sum_{i=1}^I \left(z_{b-}^i + (x_s^i \wedge 0)\right)}{\sum_{i=1}^I z_{b+}^i} \\
&= \frac{\sum_{i=1}^I \left(x_s^i + z_{b-}^i \wedge z_{b-}^i\right)}{\sum_{i=1}^I z_{b+}^i}
\end{aligned}$$

Since by Lemma (4) we know that $z_{b+}^i z_{b-}^i = 0$ in equilibrium we can conclude that

$$r_s = \frac{\sum_{i=1}^I z_{b-}^i \wedge (\omega_s^i + Y_s z_e^i)}{\sum_{i=1}^I z_{b+}^i}$$

If $\sum_{i=1}^I z_{b+}^i(q;r_{\mathbf{1}}) = 0$ the bond is not traded and $z_{b+}^i = 0$ as well as $z_{b-}^i = 0$ for all agents, hence any $r_{\mathbf{1}} \in [0,1]$ is a feasible recovery rate.

$R(r_{\mathbf{1}};q)$ is a convex valued correspondence of $[0,1]^S$ into $[0,1]^S$ which has a closed graph. By the Kakutani fixed point theorem (see Hildenbrand and Kirmann [1991]) there is a fixed point, i.e. there is a $r_{\mathbf{1}} \in R(r_{\mathbf{1}})$. Thus for any $q \in Q(r_{\mathbf{1}})$ we can always find an $r_{\mathbf{1}}$ such that $r_{\mathbf{1}} \in R(r_{\mathbf{1}})$. In particular we can therefore always find for $q^* \in Q(r_{\mathbf{1}})$ such that $G(q^*;r_{\mathbf{1}}) = 0$ a $r_{\mathbf{1}}^*$ such that $r_{\mathbf{1}}^* \in R(r_{\mathbf{1}})$ which is equivalent to $\sum_{i=1}^I \bar{\boldsymbol{x}}^{i+}(q^*;r_{\mathbf{1}}^*) = \sum_{i=1}^I \boldsymbol{\omega}^i$. It follows therefore that under assumptions [A1]-[A5] there exists a bankruptcy equilibrium. □